Varieties Generated by Truth Value Algebras of Type-2 Fuzzy Sets

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The underlying set for our algebra is $\text{Map}([0,1],[0,1])$, the set of all functions from the unit interval into itself. The operations are certain convolutions of operations on $[0,1]$, namely, $\sqcup$, $\sqcap$, $\ast$, $\bar{1}$, and $\bar{0}$ as spelled out below:

- $(f \sqcup g)(x) = \sup \{ f(y) \land g(z) : y \lor z = x \}$
- $(f \sqcap g)(x) = \sup \{ f(y) \land g(z) : y \land z = x \}$
- $f^*(x) = \sup \{ f(y) : 1 - y = x \} = f(1 - x)$
- $\bar{1}(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$
- $\bar{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

The algebra $\mathbb{M} = (\text{Map}([0,1],[0,1]),\sqcup,\sqcap,\ast,\bar{1},\bar{0})$, the algebra of truth values for fuzzy sets of type-2, is a De Morgan bisemilattice—a general algebra with two binary operations, both of which are idempotent, commutative, and associative, and an involution satisfying the De Morgan laws. The algebra $\mathbb{M}$ is not a lattice, but satisfies the equation $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$. Also note that $\mathbb{M}$ is not distributive.

**Theorem** The variety $\mathcal{V}(\mathbb{M})$ generated by $\mathbb{M}$ is generated by an algebra with 12 elements. In particular, this variety is locally finite.

This theorem is proved by showing that $\mathcal{V}(\mathbb{M})$ is generated by the subalgebra $\mathbb{E} = ([0,1],[0,1],\sqcup,\sqcap,\ast,0,1)$ of $\mathbb{M}$, and then constructing homomorphisms of $\mathbb{E}$ into an appropriate algebra with 32 elements, which in turn generates $\mathcal{V}(\mathbb{E})$. By examining congruences of this 32-element algebra, a 12-element algebra is produced which generates this 32-element algebra.

One subalgebra of $\mathbb{M}$ of interest is the algebra $\mathbb{C} = (\mathbb{C},\sqcap,\sqcup,\ast,0,1)$ consisting of the functions $f$ for which $x \leq y \leq z$ implies that $f(y) \geq f(x) \land f(z)$. One reason that this algebra is of particular interest that it is a distributive De Morgan bisemilattice.

**Theorem** The variety $\mathcal{V}(\mathbb{C})$ is generated by the four-element De Morgan algebra diamond with an absorbing element added.

For several other subalgebras and reducts of $\mathbb{M}$, we have produced finite algebras generating the same variety. A recurring theme is the role of the absorbing element, that is, the function that is identically 0. One fundamental problem remains: finding equational bases for these various varieties, especially for the algebra $\mathbb{M}$. But getting finite algebras generating these varieties is an important first step.