In the following we consider a category $\mathcal{F}$ of fuzzy sets and determine properties of this category. The objects of $\mathcal{F}$ are all pairs $(X, A)$ where $X$ is a set and $A : X \rightarrow [0, 1]$, and the maps are defined by
\[
\text{Map}_\mathcal{F}((X, A), (Y, B)) = \{ f \in \text{Map}_X (X, Y) : A(x) \leq Bf(x), \forall x \in X \}
\]
where $S$ denotes the category of sets. With ordinary composition of functions, $\mathcal{F}$ is a category.

**Proposition 1** A map $f$ in $\mathcal{F}$ is mono if and only if it is one-to-one. A map $f \in \mathcal{F}$ is epic if and only if it is onto.

**Proof.** By definition, $f$ is mono if and only if $fg = fh$ implies $g = h$. Clearly one-to-one implies $f$ is mono. Suppose $f : (X, A) \rightarrow (Y, B)$ is not one-to-one. Thus there are points $x, y \in X$ with $x \neq y$ but $f(x) = f(y)$. Let $Z = \{z\}$ be a one element set and define $g(z) = z$ and $h(z) = y$. Let $0 : Z \rightarrow [0, 1]$ be the map that takes $z$ to 0. Then $f, g, h \in \text{Map}_X ((Z, 0), (X, A))$, $fg = fh$ but $g \neq h$. Thus $f$ is not mono.

By definition, $f$ is epic if and only if $gf = hf$ implies $g = h$. Clearly $f$ onto implies $f$ is epic. Suppose $f : (X, A) \rightarrow (Y, B)$ is not onto and $g \in Y$ is not in the image of $f$. Define $g, h : Y \rightarrow Y$ by $g = \text{id}$ and $h(x) = x$ for $x \neq g$ but $h(g) = g \in \text{Im} f$. Let $0 : Y \rightarrow [0, 1]$ be the map that takes every element of $Y$ to 0. Then $g, h \in \text{Map}_X ((Y, B), (Y, 0))$, $gf = hf$, but $g \neq h$. Thus $f$ is not epic.

**Proposition 2** A map $f : (X, A) \rightarrow (Y, B)$ in $\mathcal{F}$ is an isomorphism if and only if $f$ is both one-to-one and onto and $A = Bf$.

**Proof.** Any isomorphism is both mono and epic, so by the previous Proposition, an isomorphism in $\mathcal{F}$ is both one-to-one and onto. Since $f$ is in $\mathcal{F}$, we have the inequality $A \leq Bf$, and $f^{-1}$ in $\mathcal{F}$ implies that $B \leq Af^{-1}$. It follows easily that $A = Bf$. The converse is clear.

**Proposition 3** The category $\mathcal{F}$ has products. For a family $(X_i, A_i)_{i \in J}$ the product is the pair $(X, A)$ where $X$ is the Cartesian product of the $X_i$’s and $A$ is the meet of the $A_i$’s.

**Proof.** Let $p_j : \prod_{i \in J} X_i \rightarrow X_j$ be the usual coordinate projections. Then $p_j : (\prod_{i \in J} X_i, \bigwedge_{i \in J} A_i) \rightarrow (X_j, A_j)$ satisfies
\[
\left( \bigwedge_{i \in J} A_i \right) (x) = \bigwedge_{i \in J} (A_i (x_i)) \leq A_j (p_j(x)) = A_j (x_j)
\]
so each $p_j$ is in $\mathcal{F}$. Suppose $f_j : (Y, B) \rightarrow (X_j, A_j)$ is any family of maps in $\mathcal{F}$. There is a unique function $\varphi : Y \rightarrow \prod_{i \in J} X_i$ satisfying $p_j \varphi = f_j$ . Moreover $B \leq A_j f_j$ for all $j \in J$ implies that $B \leq \left( \bigwedge_{i \in J} A_i \right) \varphi$ so $\varphi \in \mathcal{F}$. ■
Proposition 4 The category $\mathcal{F}$ has coproducts. For a family $\{ (X_i, A_i) \}_{i \in I}$ the coproduct is the pair $(X, A)$ where $X$ is the disjoint union of the $X_i$'s and $A$ is the disjoint union of the $A_i$'s.

Proof. Let $q_i : X_i \rightarrow \bigsqcup_{i \in I} X_i$ be the inclusion maps. Then $q_j : (X_i, A_i) \rightarrow (\bigsqcup_{i \in I} X_i, \bigsqcup_{i \in I} A_i)$ satisfies

$$A_j(x_j) = \left( \bigsqcup_{i \in I} A_i \right) q_j(x_j)$$

so each $q_j$ is in $\mathcal{F}$. Suppose $f_j : (X_i, A_i) \rightarrow (Y, B)$ is any family of maps in $\mathcal{F}$. There is a unique function $\varphi : \bigsqcup_{i \in I} X_i \rightarrow Y$ satisfying $\varphi q_j = f_j$. Moreover $A_j \leq B f_j$ for all $j \in J$ implies that $\bigsqcup_{i \in I} A_i \leq B \varphi$, so $\varphi \in \mathcal{F}$. ■

Proposition 5 In the category $\mathcal{F}$, the empty set with the empty map to $[0,1]$ is an initial object. A singleton set with the function taking the element to 1 in $[0,1]$ is a terminal object.

Proposition 6 The equalizer of two maps $f, g \in \text{Map}_\mathcal{F}((X, A), (Y, B))$ in the category $\mathcal{F}$ is the pair $(K, A_K)$ where $K = \{ x \in X : f(x) = g(x) \}$ and $A_K$ is the restriction of $A$ to $K$, together with the inclusion map $(K, A_K) \rightarrow (X, A)$.

Proof. Let $f, g \in \text{Map}_\mathcal{F}((X, A), (Y, B))$. Let $K = \{ x \in X : f(x) = g(x) \}$, let $i : K \rightarrow X$ be the inclusion map, and let $A_K : K \rightarrow [0,1]$ be the restriction of $A$ to $K$. Then $i \in \text{Map}_\mathcal{F}((K, A_K), (X, A))$ and we have the following diagram.

$$(K, A_K) \rightarrow (X, A) \xrightarrow{i} (Y, B)$$

Now suppose $h \in \text{Map}_\mathcal{F}((Z, C), (X, A))$ satisfies $h f = h g$. We need to show this factors uniquely through $(K, A_K)$. Clearly the image of $h$ in $K$ so there is a map $k : Z \rightarrow K$ satisfying $k h = h$. We know that $C \leq A h$. Also $A_K \leq Ah$, so $A_K = Ah$. Thus $C \leq A_K$. Then $h k \in \text{Map}_\mathcal{F}((Z, C), (K, A_K))$. Suppose $k' \in \text{Map}_\mathcal{F}((Z, C), (K, A_K))$ satisfies $k' h = h$. Then $(k' = i k = k = k' = i k = k)$. ■

Proposition 7 The coequalizer of two maps $f, g \in \text{Map}_\mathcal{F}((X, A), (Y, B))$ in the category $\mathcal{F}$ is the pair $(Z, C)$ where $Z = Y / f = g$, with $C$ the equivalence relation generated by $\{(f(x), g(x)) : x \in X\}$, and $C(z)$ is the coequalizer generated by $\{f(y) = g(z) : y \in X\}$ together with the natural projection $\pi : Y \rightarrow Z$.

Proof. By definition, $B(y) \leq C(y)$ so $\pi : (Y, B) \rightarrow (Z, C)$ is in the category $\mathcal{F}$. The set $Z$ with the map $\pi$ is the coequalizer of $f$ and $g$ in the category of sets, so for any function $h : Y \rightarrow W$ satisfying $h f = h g$, there is a unique function $\varphi : Z \rightarrow W$ satisfying $\varphi \pi = h$. Suppose that $h : (Y, B) \rightarrow (W, D)$ is a map in the category $\mathcal{F}$. We need to show that $C(z) \leq D \varphi(z)$ for all $z \in Z$. Let $y \in \pi^{-1}(z)$. We know that $B(y) \leq D h(y) = D \varphi(y) = D \varphi(z)$. Thus $C(z) = \text{lub}(B(y), \pi(y) = z) \leq D \varphi(z)$ so $\varphi$ is in the category $\mathcal{F}$. ■

2
A category is complete precisely when each family of objects has a product and each pair of parallel arrows has an equalizer, and dually for cocomplete. The category $\mathcal{F}$ is both complete and cocomplete.

**Proposition 8** The subobjects of $(X, A)$ are the pairs $(X', A')$ with $X' \subseteq X$ and $A' \leq A$.

**Note** The set of subobjects of $X$ coincides with the set of subobjects of $(X, 0)$.

**Definition 9** Define $S(X, A) = \{(X', A') : X' \subseteq X, A' \leq A\}$. For $f : (X, A) \to (Y, B)$ define $S(f) : S(X, A) \to S(Y, B)$ by $S(f)(X', A') = (f(X'), \bigvee A' f^{-1})$.

**Proposition 10** $S$ is a functor from $\mathcal{F}$ to $\mathcal{S}$.

**Proof.** First we need to show that $(f(X'), \bigvee A' f^{-1})$ is a subobject of $(Y, B)$. For this we need $\bigvee A' f^{-1} \leq B$. Let $x \in f^{-1}(y)$. Then $A'(x) \leq A(x) \leq B f(x) = B(y)$ implies that $\bigvee_{x \in f^{-1}(y)} A'(x) \leq B(y)$. Now for $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ the equality $S(g) S(f) = S(gf)$ follows from

$$\bigvee_{x \in f^{-1}(y)} A'(x) = \bigvee_{x \in g^{-1}(z) f^{-1}(y)} A'(z)$$

* Other functors from $\mathcal{F}$ to $\mathcal{S}$ are given by

  $$T(X, A) = A^{-1} [[1]]$$
  $$U(X, A) = X$$

$U$ has a left adjoint

$$V(X) = (X, 1)$$