Most developments of propositional logics follow the same general construction. First, a basic set of propositional connectives to be studied is chosen, then the set of all 'well-formed formulae' is described. At this point one has to choose the specific interpretation scheme one wants, which amounts to specifying a set of truth values and an algebraic structure on this set corresponding to the chosen connectives.

In this talk we will outline the details of this general construction using the language and concepts of Universal Algebra [2]. The universal algebraic formulation allows us to apply Birkhoff’s classical theorems about varieties of algebras to answer many questions of interest to researchers in Fuzzy Logic. For example, we can describe exactly when two choices of truth-value algebras give us the same logic, and this can help us determine finite algorithms for checking logical equivalence for a given propositional logic. In particular, we consider the propositional logic obtained when the algebra of truth values is \( \mathbb{I} \), the real numbers in the unit interval equipped with maximum, minimum, and addition, as well as negation, respectively. In this case we have the standard Fuzzy Propositional Logic. We show that this logic is equivalent to the logic obtained from choosing \( \mathbb{I} = \{0, 1, 2\} \), equipped with maximum, minimum, and the negation which fixes 1 and permutes 0 and 2, as the algebra of truth values. The propositional logic given by \( \mathbb{I} \) is called Lukasiewicz Three-valued Logic. Our result shows that these two logics are equal. That is, on the propositional logic level, there is no difference between Fuzzy Logic and Three-valued Logic. Again, another way to say this is that two propositional formulae are equivalent in Fuzzy Logic if and only if they are equivalent in Three-valued Logic.

Since logical equivalence in Three-valued logic is checkable by the use of a truth table – with 3\( ^n \) entries for checking the equivalence of two formulae involving \( n \) variables – the same is true for logical equivalence in Fuzzy Logic. Thus there is a finite algorithm for checking the logical equivalence of two formulae in Fuzzy Logic.

Both \( \mathbb{I} \) and \( \mathbb{I}^5 \) belong to the variety of De Morgan algebras. That is, the class of all algebras \( \mathbb{I} = (A, \vee, \wedge, 0, 1) \), where \( \wedge \) and \( \vee \) are commutative, associative, and idempotent, and distribute over each other; the special elements \( 0 \) and \( 1 \) are neutral and absorbent elements for \( \wedge \) and \( \vee \), and vice versa for \( \vee \) and \( \wedge \); the negation satisfies De Morgan's laws, that is, \( x^\wedge = x \cap \neg x \) and \( x^\vee = x \vee \neg x \); and \( x^\vee \wedge y = x^\wedge \vee y \) for all \( x, y \in A \). We also show, by applying classical results from Lattice Theory[1], that the only three possible propositional logics obtainable from truth-value algebras that are De Morgan algebras are: Classical Boolean Propositional Logic, Fuzzy Propositional Logic, and Interval-valued Fuzzy Propositional Logic – and no others.
The general methods we use apply not only to these special choices of truth-value algebras, but to the plethora of choices of truth value-algebras used in fuzzy systems, and so we also briefly consider what this means for propositional logics for which the algebra of truth-values is a more general DeMorgan system.

References

