Partial Orders on Type-2 Fuzzy Sets

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Description The category of fuzzy subsets has objects all functions $f : X \to [0, 1]$, and maps $\alpha : X \to Y$ satisfying $f(x) \leq g\alpha(x)$.

- This category and related ones have been studied rather thoroughly. See, for example [18], [19], [20], [21].
Definition A fuzzy subset of type-2 of a set $X$ is a function $X \rightarrow [0,1]^{[0,1]} = Map([0,1],[0,1])$, and $[0,1]^{[0,1]}$ is furnished with operations

$$(f \sqcup g)(x) = \bigvee_{y \vee z = x} (f(y) \land g(z))$$

$$(f \sqcap g)(x) = \bigvee_{y \land z = x} (f(y) \land g(z))$$

$$\overline{f}(x) = \bigvee_{y = 1 - x} f(y) = f(1 - x)$$

$$\overline{0}(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0
\end{cases} \quad \overline{1}(x) = \begin{cases} 
1 & \text{if } x = 1 \\
0 & \text{if } x \neq 1
\end{cases}$$
Definition: The algebra \([0, 1]^{[0,1]}, \sqcup, \sqcap, \neg, \tilde{0}, \tilde{1}\) is the truth value algebra of type-2 fuzzy sets.

This algebra above is due to Zadeh [1], and has been studied a great deal.

See [1], [2], [4], [5], [6], [7], [11], [12], [13], [16], and [17], for example.
The operations $\sqcap$ and $\sqcup$ may be expressed as pointwise operations on functions. For $f \in [0, 1]^{[0,1]}$, let

$$f^L(x) = \lor_{y \leq x} f(y)$$
$$f^R(x) = \lor_{y \geq x} f(y)$$

**Theorem** For $f, g \in [0, 1]^{[0,1]}$,

$$f \sqcap g = (f \lor g) \land (f^R \land g^R)$$
$$= (f \land g^R) \lor (f^R \land g)$$

$$f \sqcup g = (f \lor g) \land (f^L \land g^L)$$
$$= (f \land g^L) \lor (f^L \land g)$$
Corollary The following equations hold.

1. \( f \sqcup f = f; f \sqcap f = f \)
2. \( f \sqcup g = g \sqcup f; f \sqcap g = g \sqcap f \)
3. \( \overline{0} \sqcup f = f, \overline{1} \sqcap f = f \)
4. \( f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h; f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h \)
5. \( f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) \)
6. \( \neg \neg f = f \)
7. \( \neg (f \sqcup g) = \neg f \sqcap \neg g; \neg (f \sqcap g) = \neg f \sqcup \neg g \)

These facts may be found in [5], for example.
Proposition: Since the operations \( \cap \) and \( \cup \) are idempotent, commutative and associative, they each induce a partial order on the set \([0, 1]^{[0,1]}\) as follows:

\[
f \sqsubseteq \cap g \text{ if } f \cap g = f
\]

\[
f \sqsubseteq \cup g \text{ if } f \cup g = g
\]

The operations \( \cup \) and \( \cap \) do not give the same partial orders on \([0, 1]^{[0,1]}\). For example, \( f \sqsubseteq \cap \bar{I} \) since \( f \cap \bar{I} = f \), but it is not true that \( f \sqsubseteq \cup \bar{I} \)
Definition The category of fuzzy subsets of type-2 has objects all functions \( f : X \rightarrow [0, 1]^{[0,1]} \). A map from \( f : X \rightarrow [0, 1]^{[0,1]} \) to \( g : Y \rightarrow [0, 1]^{[0,1]} \) should be a function \( \alpha : X \rightarrow Y \) satisfying \( f(x) \leq g\alpha(x) \), where \( \leq \) is some partial order on \([0, 1]^{[0,1]}\).

\[
\begin{align*}
X & \xrightarrow{f} [0, 1]^{[0,1]} \\
\alpha & \downarrow \\
Y & \xrightarrow{g} [0, 1]^{[0,1]}
\end{align*}
\]

But we have two partial order relations at hand: \( \sqsubseteq \sqcap \) and \( \sqsubseteq \sqcup \). How do we proceed?
Solution: Investigate the partial orders $\sqsubseteq\sqcap$ and $\sqsubseteq\sqcup$.

- What properties do they have?
- Are they lattices?
- On what subalgebras are they lattices?
- When do they agree?
- What about their intersection? Is that poset a lattice?
Proposition The partial orders \( \sqcap \sqcup \) and \( \sqcap \sqcap \) each induces a semilattice on \( [0, 1]^{[0,1]} \). That is,

1. \( f \sqcap g \) in the order \( \sqcap \sqcap \) is the inf of \( f \) and \( g \).
2. \( f \sqcup g \) in the order \( \sqcap \sqcup \) is the sup of \( f \) and \( g \).
Proposition \( f \leq \sqcup g \) if and only if \( f \land g^L \leq g \leq f^L \). This is equivalent to

\[
\begin{align*}
f^L \land g &= g \\
f \land g &= f \land g^L
\end{align*}
\]

Proposition \( f \leq \sqcap g \) if and only if \( f^R \land g \leq f \leq g^R \). This is equivalent to

\[
\begin{align*}
f \land g^R &= f \\
f \land g &= f^R \land g
\end{align*}
\]
**Theorem** \([0, 1]^{[0,1], \sqcup}\) is not a lattice under the partial order \(\leq_{\sqcup}\), and \(([0, 1]^{[0,1], \sqcap}) is not a lattice under the partial order \(\leq_{\sqcap}\).

**Corollary** Infinite sup’s do not exist under \(\sqsubseteq_{\sqcup}\), and infinite inf’s do not exist under \(\sqsubseteq_{\sqcap}\).
Theorem Let $C$ be the continuous functions in $[0, 1]^{[0,1]}$. Then the subalgebra $(C, \sqcup)$ is not a lattice under the order $\leq \sqcup$. 
Remark: There are various subalgebras and subsets of $[0, 1]^{[0,1]}$ that are lattices under $\sqcup\sqcap$ or under $\sqsubseteq\sqcap$.

1. Let $S = \{f \in [0, 1]^{[0,1]} : f(0) = 1\}$. Then $(S, \sqcup, \sqcap)$ is an algebra that is a lattice under $\sqsubseteq\sqcup$.

2. Let $S = \{f \in [0, 1]^{[0,1]} : f$ is monotone increasing$\}$. Then $(S, \sqcup)$ is an algebra that is a lattice under $\sqsubseteq\sqcup$. 
**Remark** The intersection of two partial orders on a set is a partial order.

**Definition** The intersection of the partial orders $\sqsubseteq_\cap$ and $\sqsubseteq_\cup$ on $[0,1]^{[0,1]}$ is denoted $\sqsubseteq$. Thus if $f \sqsubseteq_\cap g$ and $f \sqsubseteq_\cup g$, we write $f \sqsubseteq g$. 
**Theorem** \( f \sqsubseteq g \) if and only if

\[
\begin{align*}
 f \land g^L & \leq g \leq f^L \\
 f^R \land g & \leq f \leq g^R
\end{align*}
\]

This is equivalent to

\[
\begin{align*}
 f^L \land g & = g \\
 f \land g^R & = f \\
 f \land g & = f^R \land g = f \land g^L
\end{align*}
\]

**Corollary** If \( f \sqsubseteq g \) then \( f^R \leq g^R \) and \( f^L \geq g^L \).

**Theorem** For any \( f \) and \( g \), \( (f \sqcap g) \sqsubseteq (f \sqcup g) \).
Definition For $f \in [0,1]^{[0,1]}$, the height of $f$ is $\bigvee_{x \in [0,1]} f(x)$.

1. If an element $f$ attains its height $h$, then $f$ has strong height $h$.

2. If $f$ does not attain its height $h$, then $f$ has weak height $h$.

3. The elements of height 1 are called normal, and strongly normal and weakly normal have the obvious meanings.
Proposition: Let $h \in (0, 1]$. The following hold.

1. The set of elements of weak height $h$ is a subalgebra of the reduct $([0, 1]^{[0,1]}, \cup, \cap, \neg)$. These algebras are isomorphic to each other.

2. The set of elements of strong height $h$ is a subalgebra of the reduct $([0, 1]^{[0,1]}, \cup, \cap, \neg)$. These algebras are isomorphic to each other.

3. The set of elements of height $h$ is a subalgebra of the reduct $([0, 1]^{[0,1]}, \cup, \cap, \neg)$. These algebras are isomorphic to each other.
**Theorem**  Two elements are incomparable with respect to the partial order \( \sqsubseteq \) if they have different heights.

**Corollary**  The algebra \( ([0, 1]^{[0,1]}, \sqcup, \sqcap, \neg) \) with the partial order \( \sqsubseteq \) is the disjoint union of its subalgebras of elements of height \( h, \ h \in [0, 1] \).

- Except for \( h = 0 \), these subalgebras are isomorphic to each other, so to determine what the partial order is like on these subalgebras, we may as well limit ourselves to those of height \( 1 \), the case \( h = 0 \) being trivial.
Our problem is reduced to determining the structure of the partial order \( \sqsubseteq \) on the normal functions in \([0, 1]^{[0,1]}\).

These functions are of two kinds: the convex ones and the non-convex ones.

On the normal convex ones the two partial orders given by \( \sqcap \) and \( \sqcup \) are the same.

As we will see below, that partial order has a particularly nice structure.
**Definition** A function \( f \in [0, 1]^{[0,1]} \) is **convex** if for all \( x \leq y \leq z \) in \([0, 1]\), \( f(y) \geq f(x) \land f(z) \). Equivalently, \( f = f^L \land f^R \).

**Proposition** The set of convex elements of \([0, 1]^{[0,1]}\) forms a subalgebra of \(([0, 1]^{[0,1]}), \sqcap, \sqcup, \neg\).

**Theorem** In the subalgebra of normal functions, suppose that \( f \) and \( g \) are convex. Then the \( \sup \) of \( f \) and \( g \) under the \( \sqsubseteq \) order is \( f \sqcup g \), and the \( \inf \) of \( f \) and \( g \) is \( f \sqcap g \). Also, both are convex.
**Theorem** If $f$ is convex and $g$ has height at least that of $f$, then $f \sqsubseteq f \sqcup g$.

**Theorem** If $f$ is convex and $g$ has height at least that of $f$, then $f \sqcap g \sqsubseteq f$. 
**Theorem** The set of normal convex elements of $[0, 1]^{[0,1]}$ forms a subalgebra of $([0, 1]^{[0,1]}), \sqcap, \sqcup, \neg, \bar{1}, \bar{0})$ on which the partial orders $\sqsubseteq_\sqcap$, $\sqsubseteq_\sqcup$ and $\sqsubseteq$ coincide, and this subalgebra is a complete lattice, in fact, a complete De Morgan algebra, under these common partial orders.

**Corollary** Let $C_h$ be the subalgebra of the reduct $([0, 1]^{[0,1]}), \sqcap, \sqcup, \neg$ of convex elements of height $h$. Then on $C_h$ the partial orders $\sqsubseteq_\sqcap$, $\sqsubseteq_\sqcup$ and $\sqsubseteq$ coincide, and this subalgebra is a complete lattice, in fact, a complete De Morgan algebra, under these common partial orders.
Corollary. The poset $(\mathbb{C}, \sqsubseteq)$ of convex elements of $[0, 1]^{[0,1]}$ is the disjoint union $\bigcup_h \mathbb{C}_h$ of the complete lattices $\mathbb{C}_h$.

We do not have an adequate description of the partial order $\sqsubseteq$ on the totality of the algebra of normal functions.


