Arithmetic Backwards from Shannon to the Chinese Abacus

Jerry M. Lodder*

Recall that in the 1945 white paper "First Draft of a Report on the EDVAC" (Electronic Discrete Variable Automatic Computer), John von Neumann (1903–1957) advocated the use of binary arithmetic for the digital computers of his day. Vacuum tubes afforded these machines a speed of computation unmatched by other calculational devices, with von Neumann writing: "Vacuum tube aggregates . . . have been found reliable at reaction times as short as a microsecond . . ." [7, p. 188].

Predating this, in 1938 Claude Shannon (1916–2001) published a ground-breaking paper "A Symbolic Analysis of Relay and Switching Circuits" [4] in which he demonstrated how electronic circuits can be used for binary arithmetic, and more generally for computations in Boolean algebra and logic. These relay contacts and switches performed at speeds slower than vacuum tubes. Shannon identified an economy of representing numbers electronically in binary notation as well as an ease for arithmetic operations, such as addition. These advantages of base two arithmetic are nearly identical to those cited by von Neumann. Shannon [4] writes:

> A circuit is to be designed that will automatically add two numbers, using only relays and switches. Although any numbering base could be used the circuit is greatly simplified by using the scale of two. Each digit is thus either 0 or 1; the number whose digits in order are a_k, a_{k-1}, a_{k-2}, . . ., a_2, a_1, a_0 has the value $\sum_{j=0}^{k} a_j\, 2^j$.

1. Explain how the base 10 number 95 can be written in base 2 using the above formula. In particular, compute a_k, a_{k-1}, a_{k-2}, . . ., a_2, a_1, a_0 for the number 95. What is k in this case? Write $\sum_{j=0}^{k} a_j\, 2^j$ in terms of addition symbols using the above value for k and each value of a_j.

Claude Elwood Shannon was a pioneer in electrical engineering, mathematics and computer science, having founded the field of information theory, and discovered key relationships between Boolean algebra and computer circuits [6]. Born in the state of Michigan in 1916, he showed an interest in mechanical devices, and studied both electrical engineering and mathematics at the University of Michigan. Having received Bachelor of Science degrees in both of these subjects, he then accepted a research assistantship in the Department of Electrical Engineering at the Massachusetts Institute of Technology. The fundamental relation between Boolean logic and electrical circuits formed the topic of his master's thesis at MIT, and became his first published paper [4], for which he received an award from the combined engineering societies of the United States. In 1940 he earned his doctorate in mathematics from MIT with the dissertation "An Algebra for Theoretical Genetics." He spent the academic year 1940–41 visiting the Institute for Advanced Study in Princeton, New Jersey, where he began to formulate his ideas about information theory and efficient communication systems. The next fifteen years were productively spent at Bell Laboratories, and in 1948 he launched a new field of study, information theory, with the paper "A Mathematical Theory of Communication" [5]. He published extensively in communication theory, cryptography,

*Mathematical Sciences; Dept. 3MB, Box 30001; New Mexico State University; Las Cruces, NM 88003; jlodder@nmsu.edu.

game theory and computer science. In 1956 Dr. Shannon accepted a professorship at MIT, and retired in 1978.

Let's read a few excerpts from "A Symbolic Analysis of Relay and Switching Circuits" [4] with an eye toward understanding the circuitry behind binary arithmetic.

A Symbolic Analysis of Relay
and Switching Circuits

Claude E. Shannon

I. Introduction

In the control and protective circuits of complex electrical systems it is frequently necessary to make intricate interconnections of relay contacts and switches. Examples of these circuits occur in automatic telephone exchanges, industrial motor-control equipment, and in almost any circuits designed to perform complex operations automatically. In this paper a mathematical analysis of certain of the properties of such networks will be made. . . .

II. Series-Parallel Two-Terminal Circuits

Fundamental Definitions and Postulates

We shall limit our treatment of circuits containing only relay contacts and switches, and therefore at any given time the circuit between two terminals must be either open (infinite impedance) or closed (zero impedance). Let us associate a symbol X_{ab} or more simply X with the terminals a and b. This variable, a function of time, will be called the hindrance of the two-terminal circuit a–b. The symbol 0 (zero) will be used to represent the hindrance of a closed circuit and the symbol 1 (unity) to represent the hindrance of an open circuit. Thus when the circuit a–b is open $X_{ab} = 1$ and when closed $X_{ab} = 0$. . . . Now let the symbol $+$ (plus) be defined to mean the series connection of the two-terminal circuits whose hindrances are added together. Thus $X_{ab} + X_{cd}$ is the hindrance of the circuit a–d when b and c are connected together. Similarly the product of two hindrances $X_{ab} \cdot X_{cd}$ or more briefly $X_{ab}X_{cd}$ will be defined to mean the hindrance of the circuit formed by connecting the circuits a–b and c–d in parallel. A relay contact or switch will be represented in a circuit by the symbol in Figure 1, the letter being the corresponding hindrance function. Figure 2 shows the interpretation of the plus sign and Figure 3 the multiplication sign.

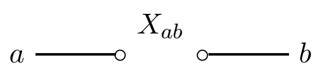


Figure 1. Symbol for hindrance function.

Figure 2. Interpretation of addition.

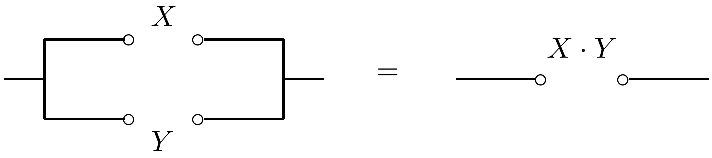
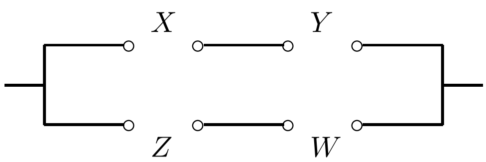


Figure 3. Interpretation of multiplication.

2. Consider X_{ab} as a Boolean variable with only two possible values 0 or 1. If $X_{ab} = 0$, then the switch in Figure 1 is closed, and current flows from a to b. If $X_{ab} = 1$, then the switch is open, and current does not flow from a to b. Now let X and Y be two Boolean variables. In a table listing all possible values for X and Y, record the results for X and Y joined in series, i.e., $X + Y$. Be sure to justify your answer by discussing whether current flows in the series circuit. Note that if current flows from left to right in Figure 2, then $X + Y = 0$, while if current does not flow, then $X + Y = 1$. To what extent does $X + Y$ represent a usual notation of addition? To what extent does $X + Y$ represent a construction in logic?

3. In a table listing all possible values for X and Y, record the results for X and Y joined in parallel, i.e., $X \cdot Y$. Justify your answer by discussing whether current flows in the parallel circuit. To what extent does $X \cdot Y$ represent a usual notion of multiplication? To what extent does $X \cdot Y$ represent a construction in logic?

4. These basic series and parallel circuits may be combined in any combination, using any finite number of switches. let X, Y, Z and W be Boolean variables, used as switches in the picture below. In a table listing all possible values for X, Y, Z and W, compute the value (0 or 1) of the circuit:



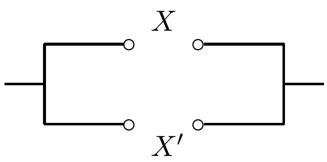
Explain your result in terms of a simple logical construction involving the results for the basic circuits $X + Y$ and $Z + W$:

Shannon continues:

> We shall now define a new operation to be called negation. The negative of a hindrance X will be written X' and is defined to be a variable which is equal to 1 when X equals 0 and equal to 0 when X equals 1.

5. In a table listing both values for X, compute the value of the circuit:

In another table, compute the value of the circuit:



Can you interpret these tables in terms of simple logical constructions?

6. Now let a and b be binary variables with one digit. Recall from the project "Binary Arithmetic: From Leibniz to von Neumann" that the digit in the ones place (the right-hand digit) of the *arithmetic sum* $a + b$ can be expressed using the "exclusive or" operation. Find a circuit which gives this digit. Justify your answer with a table that lists all possible values of a and b. Note that the arithmetic sum $a + b$ is the result of adding the binary values of a and b, not a and b combined in series.

7. Let a and b be binary variables with one digit as in question 6. Find a circuit which gives the digit in the twos place (the left-hand digit) of the arithmetic sum $a + b$. Justify your answer using a table listing all possible values of a and b.

8. Let a and b be binary variables with two possible digits. In Shannon's notation,

$$a = a_1 2^1 + a_0 2^0, \quad b = b_1 2^1 + b_0 2^0.$$

The digits of a are a_1, a_0, and the digits of b are b_1, b_0. Let c_0 be the result of the carried digit from $a_0 + b_0$. Either $c_0 = 0$ or $c_0 = 1$. In terms of the values for a_1, b_1 and c_0, when is the digit in the twos place for the arithmetic sum $a + b$ equal to zero? equal to one? Using a_1, b_1 and c_0 as switches, find a circuit which gives the digit in the twos place for $a + b$. Justify your answer using a table that lists all possible values for a_1, b_1 and c_0.

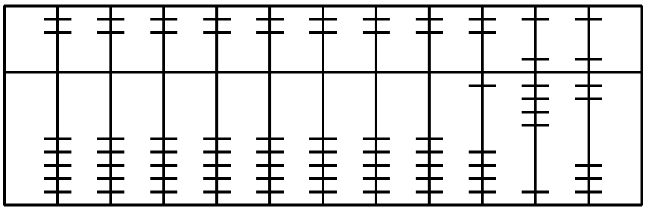
9. Let a and b be binary variables with two digits as in part question 8. Using a_1, b_1 and c_0 as switches, find a circuit which gives the digit in the fours place (the left-most digit) of $a + b$. Justify your answer using a table listing all possible values for a_1, b_1 and c_0.

Recall that of binary numeration, Leibniz [1, p. 225] writes:

> However, I am not at all recommending this manner of counting as a replacement for the ordinary practice of tens. . . . [The] practice of tens is shorter, the numbers not as long. If we were accustomed to proceed by twelves or by sixteens, there would be even more of an advantage.

Writing large numbers by hand in binary notation easily results in transcription errors, since there are often many digits in a number base 2. However, entering base 10 numbers on a computer requires a conversion to base 2 at some level, a conversion which is not readily made, since 10 is not an integral power of 2. Let's now examine the Chinese abacus, and remember that Leibniz's intellectual curiosity had led him to the study of Chinese culture and religion, with an interpretation of the ancient text *Yijing* (the *I-ching* or *Book of Changes*) in terms of binary numeration.

The Chinese abacus (*suan pan*) consists of bars set in a rectangular frame, with the number of bars being 9, 11, 13, 17 or more [2, p. 211]. Each bar contains two upper beads and five lower beads, separated by a crossbar. Each upper bead counts as five units, and each lower bead counts as one unit. Traditionally numbers are represented positionally using base 10. A decimal point could be arbitrarily chosen between two bars of the abacus, and the digits are then arranged from left to right in decreasing powers of ten, so that the ones place is to the right of the tens place, the tens place is to the right of the hundreds place [3, p. 74–75]. Before representing a number on the abacus, all beads are moved away from the central crossbar so that they rest against the frame. Placing the decimal point at the far right of the frame, the base 10 number 197 would be displayed by moving one upper bead and two lower beads against the crossbar of the right-most bar, one upper bead and four lower beads against the crossbar of the bar immediately to the left of that, and one lower bead against the crossbar of the next bar.



The number 197 set on a Chinese abacus.

Go to the library or the world wide web and research how (base 10) addition is performed on an abacus. Pay particular attention to the operation known today as “carrying.” Notice that representing a number in base 10 requires a minimum of ten distinct values for bead arrangements along a given bar, which includes the value zero.

10. On a Chinese abacus, how many distinct numerical values can be represented along a given bar, including the value zero? Note that certain numbers greater than ten can be constructed by using two five beads on the same bar. Let N denote this number of distinct values. If the full range of values for bead arrangements is employed on each bar, what number base is represented on a Chinese abacus?

11. Using N as in question 10, write the base 10 numbers from 1 to 20 inclusive in base N. Although you may invent any symbols that you wish for additional digits in this new base, be sure to explain what your symbols mean. For now, call the new digits n_1, n_2, n_3, etc., where $n_1 = 10$, $n_2 = 11$, $n_3 = 12$, etc.

12. In a table list the base 10 numbers 1 through 32 inclusive, their binary equivalents and their base N equivalents. Is there a pattern between the base 2 and base N representations? Explain.

13. Consider the number $9n_2 5n_4$ in base N, where n_4 is in the ones place, 5 in the N's place, n_2 in the N^2's place, and 9 in the N^3's place. Explain how to perform the addition

$$9n_2 5n_4 + n_1 72n_2$$

in base N on a Chinese abacus. What is the value of this sum in base N? Convert the final sum to base 10, and explain the conversion process.

Extra Credit: What is the sum $9n_2 5n_4 + n_1 72n_2$ in base 2? Justify your answer.

Notes to the Instructor

The project builds naturally on the previous offering “Binary Arithmetic: From Leibniz to von Neumann,” and is well suited for a first course in discrete mathematics or computer science. The project also includes an examination of arithmetic on a Chinese abacus, and, in a departure from the historical record, explores base sixteen (hexadecimal) operations on the abacus. This use results from the full potential of all numerical values that can be represented on one bar of the Chinese abacus, and provides as an enrichment exercise in two-power base arithmetic. Today base sixteen is often used by computer scientists as a shorthand for base two, since, as observed by Leibniz, larger bases afford shorter lengths of notation in place value numeration.

References

[1] Gerhardt, C. I., (editor) *G. W. Leibniz Mathematische Schriften*, Vol. VII, Olms, Hildesheim, 1962.

[2] Martzloff, J.-L., *A History of Chinese Mathematics*, Wilson, S.S. (translator), Springer Verlag, Berlin, 1997.

[3] Needham, J., *Science and Civilisation in China*, vol. 3, Cambridge University Press, Cambridge, 1959.

[4] Shannon, C., "A Symbolic Analysis of Relay and Switching Circuits," *Transactions American Institute of Electrical Engineers*, **57** (1938), 713–723.

[5] Shannon, C.E., "A Mathematical Theory of Communication," *Bell System Technical Journal*, **27** (1948), 379–423 and 623–656.

[6] Sloane, N.J.A., Wyner, A.D. (editors), *Claude Elwood Shannon: Collected Papers*, The Institute of Electrical and Electronics Engineers, Inc., New York, 1993.

[7] von Neumann, J., "First Draft of a Report on the EDVAC," in *From ENIAC to UNIVAC: An Appraisal of the Eckert-Mauchly Computers*, N. Stern, Digital Press, Bedford, Massachusetts, 1981, 177–246.