Some Comments on Interval Valued Fuzzy Sets*

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Abstract

This paper presents a framework for fuzzy set theory in which fuzzy values are intervals.

1 Introduction

Let S be a set. A fuzzy subset of S is a mapping $A : S \to [0,1]$. The value $A(x)$ for a particular x is typically associated with a degree of belief of some expert. There is now a quite extensive theory of fuzzy sets, the basics involving putting operations on the set $\mathcal{F}(S)$ of all fuzzy subsets of the set S. These operations stem from operations on the unit interval $[0,1]$, which has a rather elaborate structure. First, $[0,1]$ is a complete distributive lattice with respect to the order on it. In addition, it is equipped with the usual arithmetic operations of real numbers. Endless combinations of lattice and arithmetic operations make possible a host of operations on $[0,1]$ and hence on $\mathcal{F}(S)$. But an increasingly prevalent view is that models based on $[0,1]$ are inadequate. Many believe that assigning an exact number to an expert's opinion is too restrictive, and that the assignment of an interval of values is more realistic. We will outline the basic framework of models in which fuzzy values are intervals. The reference [1] provides a background for this paper, both in point of view and in notation and terminology.

2 Interval Valued Fuzzy Sets

We take the following point of view. For ordinary fuzzy set theory, the basic structure on $[0,1]$ is its lattice structure, coming from its order $\leq$. The interval $[0,1]$ is a lattice, and this entity $([0,1], \leq)$ is denoted $\mathbb{I}$. Subsequent operations on $[0,1]$ are operations on $\mathbb{I}$. For example, t-norms on $[0,1]$ are required to behave in special ways with respect to $\leq$. An elaboration of this point of view is in [1]. In any case, the underlying structure of the whole theory is the mathematical entity $\mathbb{I}$.

Now consider fuzzy sets with interval values. The interval $[0,1]$ is replaced by the set $\{(a,b) : a, b \in [0,1],\ a \leq b\}$. The element (a,b) is just the *pair* with $a \leq b$. In lattice theory, there is a standard notation for this set: $[0,1]^{[2]}$. So if S is the universal set, then our new fuzzy sets are mappings $A : S \to [0,1]^{[2]}$. Now comes the crucial question. With what structure should $[0,1]^{[2]}$ be endowed? Again, lattice theory provides an answer. Use

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componentwise operations coming from the operations on $[0,1]$. For example, $(a,b) \leq (c,d)$ if $a \leq c$ and $b \leq d$, which gives the usual lattice max and min operations

$$\begin{aligned} (a,b) \vee (c,d) &= (a \vee c, b \vee d) \\ (a,b) \wedge (c,d) &= (a \wedge c, b \wedge d) \end{aligned}$$

The resulting structure again has the standard notation $\mathbb{I}^{[2]}$. That is, $\mathbb{I}^{[2]}$ is the set $[0,1]^{[2]}$ with componentwise operations. This is a fundamental lattice theoretical construction: from a lattice L, form $L^{[2]}$ and use componentwise operations. The resulting lattice has many of the same properties as the original lattice. In particular, if L is a complete distributive lattice, then so is $L^{[2]}$. The proposal here is to use $\mathbb{I}^{[2]}$ as the basic building block for interval valued fuzzy set theory.

There is a natural negation on $\mathbb{I}^{[2]}$ given by $(a,b)' = (b', a')$ where $x' = 1 - x$. With this negation, $\mathbb{I}^{[2]}$ becomes a DeMorgan algebra $(\mathbb{I}^{[2]}, ', 0, 1)$. This in turn yields a DeMorgan algebra for the set of all interval valued fuzzy sets with the operations

$$\begin{aligned} (A \wedge B)(s) &= A(s) \wedge B(s) \\ (A \vee B)(s) &= A(s) \vee B(s) \\ A'(s) &= (A(s))' \end{aligned}$$

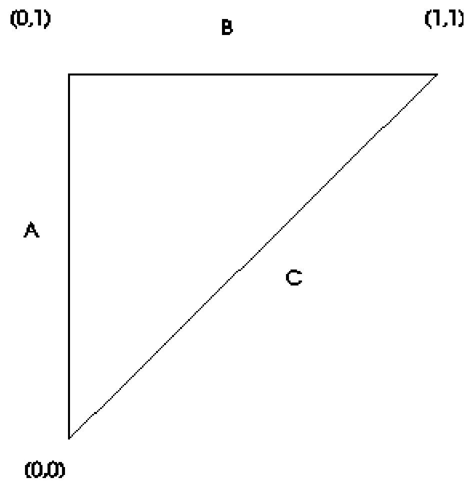
3 Automorphisms

A good share of the theory of fuzzy sets is concerned with endowing $\mathbb{I}$ with additional structure such as t-norms, t-conorms, and negations other than the usual min, max, and the negation $x \rightarrow 1 - x$. We address the situation for $\mathbb{I}^{[2]}$ and a basic problem will be deciding on the appropriate definitions. For example, what should a t-norm on $\mathbb{I}^{[2]}$ be? Before addressing that problem, we first examine the set of automorphisms and anti-automorphisms of $\mathbb{I}^{[2]}$. This is fundamental for representation theorems for certain norms, conorms, and negations. We have no choice about what automorphisms and anti-automorphisms are: they are automorphisms and anti-automorphisms of our basic structure $\mathbb{I}^{[2]}$.

Definition 1 *An* ***automorphism*** *of* $\mathbb{I}^{[2]}$ *is a one-to-one map* f *from* $[0,1]^{[2]}$ *onto itself such that* $f(x) \leq f(y)$ *if and only if* $x \leq y$*. An* ***anti-automorphism*** *is a one-to-one map* f *from* $[0,1]^{[2]}$ *onto itself such that* $f(x) \leq f(y)$ *if and only if* $x \geq y$*.*

The set of all automorphisms of $\mathbb{I}^{[2]}$ is denoted $Aut(\mathbb{I}^{[2]})$ and the set of all automorphisms and anti-automorphisms is denoted $Map(\mathbb{I}^{[2]})$. Both of these are groups under composition of maps, and $Aut(\mathbb{I}^{[2]})$ is a normal subgroup of index 2 in $Map(\mathbb{I}^{[2]})$. Except for the identity, all the elements of $Aut(\mathbb{I}^{[2]})$ are of infinite order. Anti-automorphisms are of order two or of infinite order. These are trivial but pertinent facts. Anti-automorphisms of order two are **involutions,** and that set is denoted $Inv(\mathbb{I}^{[2]})$. The map α given by $\alpha(a,b) = (1-b, 1-a)$ is an involution, as is $f^{-1}\alpha f$ for any $f \in Map(\mathbb{I}^{[2]})$. It turns out that there are no others. If f is an automorphism of $\mathbb{I}$, that is, a one-to-one map of $[0,1]$ onto itself such that $f(x) \leq f(y)$ if and only if $x \leq y$, then $(a,b) \rightarrow (f(a), f(b))$ is an automorphism of $\mathbb{I}^{[2]}$. It turns out that there are no others. It should be clear that automorphisms take $(0,0)$ and $(1,1)$ to themselves. Anti-automorphisms interchange these two elements.

In the plane, $[0,1]^{[2]}$ is the triangle pictured. Each leg is mapped onto itself by automorphisms.



Lemma 1 *Let* $A = \{(0,x) : x \in (0,1)\}$, $B = \{(x,1) : x \in (0,1)\}$, *and* $C = \{(x,x) : x \in (0,1)\}$. *If* $f \in Aut(\mathbb{I}^{[2]})$ *then* $f(A) = A$, $f(B) = B$, *and* $f(C) = C$.

Proof. Let $f \in Aut(\mathbb{I}^{[2]})$. Since $f(x) \leq f(y)$ if and only if $x \leq y$, it follows that

$$\begin{aligned} f(x \vee y) &= f(x) \vee f(y) \\ f(x \wedge y) &= f(x) \wedge f(y) \\ f(0,0) &= (0,0) \\ f(1,1) &= (1,1) \end{aligned}$$

Suppose that $f(a,b) = (c,c)$. Then since $(a,b) = (a,a) \vee ((b,b) \wedge (0,1))$,

$$\begin{aligned} f(a,b) &= f((a,a) \vee ((b,b) \wedge (0,1)) \\ &= f(a,a) \vee (f(b,b) \wedge f(0,1)) \end{aligned}$$

and thus

$$f(a,a) \vee (f(b,b) \wedge f(0,1)) = (c,c).$$

No two elements strictly less than a diagonal element (c,c) can have join (c,c) and no two elements strictly greater than a diagonal element (c,c) can have meet (c,c). Thus (c,c) is equal to one of $f(a,a)$, $f(b,b)$, and $f(0,1)$, from which it follows from the injectivity of f that (a,b) is equal to one of (a,a), (b,b), or $(0,1)$. We need only rule out $f(0,1) = (c,c)$. So suppose that $f(0,1) = (c,c)$. Then every element less than (c,c) is the image of an element less than $(0,1)$. Since the elements less than $(0,1)$ form a chain, this implies that the elements less than (c,c) form a chain, which is not the case unless $c = 0$. But then $f(0,1) = (0,0) = f(0,0)$ which violates the injectivity of f. Therefore only elements from C go to C under an automorphism f. Since the inverse of an automorphism is an automorphism, $f(C) = C$ for all automorphisms f.

If $f(a,b) = (0,c)$, then

$$\begin{aligned} f(a,b) &= f(a,a) \vee f(0,b) \\ &= (d,d) \vee f(0,b) \\ &= (0,c) \end{aligned}$$

Therefore $d = 0$, implying that $a = 0$. Hence $f(A) = A$.

If $f(a,b)=(c,1)$, then

$$\begin{aligned} f(a,b) &= f(b,b)\wedge f(a,1) \\ &= (d,d)\wedge f(a,1) \\ &= (c,1) \end{aligned}$$

Thus $d=1$, and so $b=1$. It follows that $f(B)=B$. □

Theorem 2 *Every automorphism f of $\mathbb{I}^{[2]}$ is of the form $f(a,b)=(g(a),g(b))$, where g is an automorphism of $\mathbb{I}$.*

Proof. Since f is an automorphism of C, it induces an automorphism g of $\mathbb{I}$, namely $(g(x),g(x))=f(x,x)$. Now $f(0,1)=(0,1)$ since $f(A)=A$ and $f(B)=B$. Thus

$$\begin{aligned} f(a,b) &= f(a,a)\vee(f(b,b)\wedge f(0,1)) \\ &= (g(a),g(a))\vee((g(b),g(b))\wedge(0,1)) \\ &= (g(a),g(b)) \end{aligned}$$

□

So automorphisms of $\mathbb{I}^{[2]}$ are of the form $(a,b)\to(f(a),f(b))$ where f is an automorphism of $\mathbb{I}$. We will use f to denote both the automorphism of $\mathbb{I}$ and the corresponding automorphism of $\mathbb{I}^{[2]}$. The following theorem is clear.

Theorem 3 $Aut(\mathbb{I})\approx Aut(\mathbb{I}^{[2]})$.

Let α be the anti-automorphism of $\mathbb{I}$ given by $\alpha(a)=1-a$. Then $(a,b)\to(\alpha(b),\alpha(a))$ is an anti-automorphism of $\mathbb{I}^{[2]}$ which we also denote by α. If g is an anti-automorphism of $\mathbb{I}^{[2]}$, then $g=\alpha f$ for the automorphism $f=\alpha g$. Now

$$\begin{aligned} g(a,b) &= \alpha f(a,b) \\ &= \alpha(f(a),f(b)) \\ &= (\alpha f(b),\alpha f(a)) \\ &= (g(b),g(a)) \end{aligned}$$

We have

Corollary 4 *Every anti-automorphism of $\mathbb{I}^{[2]}$ is of the form $(a,b)\to(g(b),g(a))$, where g is an anti-automorphism of $\mathbb{I}$.*

Theorem 5 $Map(\mathbb{I})\approx Map(\mathbb{I}^{[2]})$.

A lot of facts about automorphisms and anti-automorphisms now follow from the facts in [1]. In particular, we have

Theorem 6 *The involutions of $\mathbb{I}^{[2]}$ are precisely the involutions $\{f^{-1}\alpha f : f\in Aut(\mathbb{I}^{[2]})\}$.*

4 t-norms

Our first problem here is that of *defining* t-norms. There is a natural embedding of $\mathbb{I}$ into $\mathbb{I}^{[2]}$, namely $c \to (c, c)$. This is how $\mathbb{I}^{[2]}$ generalizes $\mathbb{I}$. Instead of specifying a number c (identified with (c, c)) as a degree of belief, an expert specifies an interval (a, b) with $a \leq b$. So no matter how a t-norm is defined on $\mathbb{I}^{[2]}$, it should induce a t-norm on this copy of $\mathbb{I}$ in it.

A t-norm should be increasing in each variable, just as in the case for $\mathbb{I}$. On $\mathbb{I}$ this is equivalent to the conditions $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$. But just increasing in each variable will not yield these distributive laws on $\mathbb{I}^{[2]}$. However, these distributive laws do imply increasing in each variable.

Now for the boundary conditions. The two elements $(0, 0)$ and $(1, 1)$ are the bounds of the lattice $\mathbb{I}^{[2]}$. These two elements, along with $(0, 1)$, play a special role in the lattice, as they are the only elements of the lattice fixed by all automorphisms. Since t-norms on $\mathbb{I}^{[2]}$ are to generalize t-norms on $\mathbb{I}$, we certainly want $(1, 1) \circ (a, b) = (a, b)$ for all $(a, b) \in \mathbb{I}^{[2]}$. It will follow that $(0, 0) \circ (a, b) = (0, 0)$, but what about the element $(0, 1)$? How is it to behave? In analogy, it is natural to require that $(0, 1) \circ (a, b) = (0, b)$. We are led to the following definition.

Definition 2 *A commutative, associative binary operation $\circ$ on $\mathbb{I}^{[2]}$ is a **t-norm** if for all $x, y, z \in [0, 1]^{[2]}$ and $(a, b) \in [0, 1]^{[2]}$*

1. *$C \circ C \subseteq C$, where $C = \{(c, c) : c \in [0, 1)\}$*
2. *$x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$*
3. *$x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$*
4. *$(1, 1) \circ x = x$*
5. *$(0, 1) \circ (a, b) = (0, b)$.*

Several additional useful properties follow immediately for a t-norm $\circ$ on $\mathbb{I}^{[2]}$.

1. $\circ$ is increasing in each variable.
2. $x \circ y \leq x \wedge y$
3. $(0, 0) \circ x = (0, 0)$
4. $(0, b) \circ x = (0, e)$ for some e.
5. The restriction of $\circ$ to A, B, or C induces a t-norm on $\mathbb{I}$.

To see that $\circ$ is increasing, suppose that $y \leq z$. Then

$$x \circ z = x \circ (y \vee z) = (x \circ y) \vee (x \circ z).$$

For the fourth property, note that $(0, b) \circ (c, d) \leq (0, 1) \circ (c, d) = (0, d)$, and observe that all elements less than or equal to $(0, d)$ are of the form $(0, e)$ for some e.

We can now show that t-norms on $\mathbb{I}^{[2]}$ are a natural extension of t-norms on $\mathbb{I}$.

Theorem 7 *Every t-norm $\diamond$ on $\mathbb{I}^{[2]}$ is of the form*

$$(a, b) \diamond (c, d) = (a \circ c, b \circ d)$$

where $\circ$ is a t-norm on $\mathbb{I}$.

Proof. Since the t-norm induces a t-norm on C, we have

$$(a,a) \diamond (c,c) = (a \circ c, a \circ c)$$

where $\circ$ is a t-norm on $\mathbb{I}$. Now

$$\begin{aligned}(a,b) \diamond (c,d) &= (a,b) \diamond ((c,c) \vee (0,d)) \\ &= (a,b) \diamond (c,c) \vee (a,b) \diamond (0,d) \\ &= (a,b) \diamond (c,c) \vee (0,e)\end{aligned}$$

Therefore, the first component of $(a,b) \diamond (c,d)$ does not depend on d and similarly does not depend on b. Also $(a,b) \diamond (c,d)$ has second component the same as

$$\begin{aligned}(a,b) \diamond (c,d) \diamond (0,1) &= (a,b) \diamond (0,1) \diamond (c,d) \diamond (0,1) \\ &= (0,b) \diamond (0,d)\end{aligned}$$

Thus the second component of $(a,b) \diamond (c,d)$ does not depend on a or c. So a t-norm on $\mathbb{I}^{[2]}$ acts componentwise. From $(a,a) \diamond (c,c) = (a \circ c, a \circ c)$ it follows that

$$(a,b) \diamond (c,d) = (a \circ c, b \circ d)$$

and the proof is complete. □

Most of the t-norms we will consider are convex in the following sense. The convex t-norms on $\mathbb{I}$ or $\mathbb{I}^{[2]}$ are exactly the continuous t-norms.

Definition 3 *A binary operation $\circ$ on $\mathbb{I}$ or $\mathbb{I}^{[2]}$ is* ***convex*** *if given $x \circ y \leq c \leq u \circ v$, there exists an r between x and u and an x between y and v such that $c = r \circ s$.*

In the case of convex binary operations we can weaken condition 5 of the definition of t-norm.

Theorem 8 *A commutative, associative, convex binary operation on is a t-norm if for all $x, y, z \in [0,1]^{[2]}$*

1. *$C \circ C \subseteq C$, where $C = \{(c,c) : c \in [0,1)\}$*
2. *$x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$*
3. *$x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$*
4. *$(1,1) \circ x = x$*
5. *$(0,1) \circ (0,1) = (0,1)$.*

Proof. Suppose $\circ$ is a commutative, associative, convex binary operation on $\mathbb{I}^{[2]}$. Then for any $x \in \mathbb{I}^{[2]}$, $x \circ 0 = (x \wedge 1) \circ 0 = (x \circ 0) \wedge (1 \circ 0) = (x \circ 0) \wedge 0 = 0$. If $x \circ x = x$, then for any $y \in \mathbb{I}^{[2]}$, $x \circ 0 = 0 \leq x \wedge y \leq x = x \circ x$, so there is an element z with $0 \leq z \leq x$ and $x \circ z = x \wedge y$. Then

$$x \wedge y = x \circ z = (x \circ x) \circ z = x \circ (x \circ z) = x \circ (x \wedge y) \leq x \circ y.$$

But $x \circ y = x \circ (y \wedge 1) = (x \circ y) \wedge (x \circ 1) \leq x$ and $x \circ y = (x \wedge 1) \circ y = (x \circ y) \wedge (1 \circ y) \leq y$. Thus $x \circ y = x \wedge y$. Now, since $(0,1) \circ (0,1) = (0,1)$, we have

$$(0,1) \circ (a,b) = (0,1) \wedge (a,b) = (0,b).$$

□

Now we define Archimedean, strict, and nilpotent t-norms on $\mathbb{I}^{[2]}$ just as for t-norms on $\mathbb{I}$. In the context of a t-norm $\circ$ we will write $x^n = \overbrace{x \circ x \circ \cdots \circ x}^{n \text{ times}}$.

Definition 4 *A t-norm $\circ$ on $\mathbb{I}^{[2]}$ is **Archimedean** if given any $x, y \in \mathbb{I}^{[2]}$ with $x, y \notin \{(0,0),(0,1),(1,1)\}$, there is a positive integer n with $x^n \leq y$. A convex Archimedean t-norm is **strict** if $x \circ x = 0$ only for $x = 0$ and **nilpotent** otherwise.*

The characterization of convex (continuous) Archimedean t-norms on $\mathbb{I}^{[2]}$ is analogous to that for $\mathbb{I}$.

Proposition 9 *A convex t-norm $\circ$ on $\mathbb{I}^{[2]}$ is Archimedean if and only if it satisfies $x \circ x < x$ for all $x \in \mathbb{I}^{[2]} \backslash \{(0,0),(0,1),(1,1)\}$.*

A convex Archimedean t-norm on $\mathbb{I}$ is nilpotent if and only if for each $x \in [0,1]$, there is a positive integer n such that $x^n \in \{0,1\}$. The corresponding condition for a nilpotent convex Archimedean t-norm on $\mathbb{I}^{[2]}$ is that for each $x \in [0,1]^{[2]}$ there is a positive integer n such that $x^n \in \{(0,0),(0,1),(1,1)\}$.

It is easy to see that, in the notation of Theorem 7, a convex t-norm $\diamond$ is Archimedean, strict, or nilpotent if and only if the t-norm $\circ$ is Archimedean, strict, or nilpotent, respectively. In effect, the theory of t-norms on $\mathbb{I}^{[2]}$ as we have defined them is reduced to the theory of t-norms on $\mathbb{I}$.

5 Negations and t-conorms

An anti-automorphism f such that $f(f(x)) = x$ is an **involution**, or **negation**. The map α given by $\alpha(a,b) = (1-b, 1-a)$ is a negation, as is $f^{-1}\alpha f$ for any automorphism f, and there are no others. Anti-automorphisms interchange $(0,0)$ and $(1,1)$. Just as for $\mathbb{I}$, we define a t-conorm to be the dual of a t-norm with respect to some negation.

Definition 5 *Let $\triangle$ be a binary operation and η a negation on $\mathbb{I}^{[2]}$. The **dual of** $\triangle$ **with respect to** η is the binary operation $\triangledown$ given by*

$$a \triangledown b = \eta(\eta(a) \triangle \eta(b))$$

*If $\triangle$ is a t-norm, then $\triangledown$ is called a **t-conorm**.*

Theorem 10 *Every t-conorm $\triangledown$ on $\mathbb{I}^{[2]}$ is of the form*

$$(a,b) \triangledown (c,d) = (a \triangledown c, b \triangledown d)$$

where $\triangledown$ is a t-conorm on $\mathbb{I}$.

The proof follows by duality from the proof for t-norms. Thus the theory of t-conorms and negations on $\mathbb{I}^{[2]}$ has also been reduced to that theory on $\mathbb{I}$.

The following theorem gives properties characterizing t-conorms.

Theorem 11 *A commutative, associative binary operation $\triangledown$ on $\mathbb{I}^{[2]}$ is a t-conorm if and only if for all $x, y, z, (a,b) \in \mathbb{I}^{[2]}$*

1. *$C \triangledown C = C$, where $C = \{(x,x) : x \in \mathbb{I}\}$*

2. $x \triangledown (y \vee z) = (x \triangledown y) \vee (x \triangledown z)$

3. $x \triangledown (y \wedge z) = (x \triangledown y) \wedge (x \triangledown z)$

4. $(0,0) \triangledown x = x$

5. $(0,1) \triangledown (a,b) = (a,1)$.

Definition 6 *A t-conorm $\triangledown$ is* ***convex*** *if given $x \triangledown y \leq c \leq u \triangledown v$, there exists an r between x and u and an s between y and v such that $c = r \triangledown s$.*

Theorem 12 *Every convex t-conorm $\diamond$ on $\mathbb{I}^{[2]}$ is of the form*

$$(a,b) \diamond (c,d) = (a \triangledown c, b \triangledown d)$$

where $\triangledown$ is a t-conorm on $\mathbb{I}$.

Note that for a convex binary operation $\triangledown$ on $\mathbb{I}$ or $\mathbb{I}^{[2]}$, if $x \triangledown x = x$, then $x \triangledown y = x \vee y$ for all y. It follows that for a convex binary operation, condition 5 of the characterization of t-conorms can be replaced by the weaker condition $(0,1) \triangledown (0,1) = (0,1)$.

Implications are defined in terms of t-norms, t-conorms, and negations, so one can also develop the theory of implications for $\mathbb{I}^{[2]}$ from that of $\mathbb{I}$. In conclusion, the theory of DeMorgan systems on $\mathbb{I}^{[2]}$ is a natural extension of the theory of DeMorgan systems on $\mathbb{I}$.

References

[1] Gehrke, M., Walker, C., and Walker, E., DeMorgan systems on the unit interval, *Int. Jour. Intelligent Systems*, to appear.