INTRODUCTION
TO ABSTRACT
ALGEBRA

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Preface

In teaching a beginning course in abstract algebra, one must suppress the urge to cover a lot of material and to be as general as possible. The difficulties in teaching such a course are pedagogical, not mathematical. The subject matter is abstract, yet it must be kept meaningful for students meeting abstractness for perhaps the first time. It is better for a student to know what a theorem says than to be able to quote it, produce a proof of it detail by detail, and not have the faintest notion of what it’s all about. However, careful attention must be paid to rigor, and sloppy thinking and incoherent writing cannot be tolerated. But rigor should be flavored with understanding. Understanding the content of a theorem is as important as being able to prove it. I have tried to keep these things in mind while writing this book.

The specific subject matter chosen here is standard, and the arrangement of topics is not particularly bizarre. In an algebra course, I believe one should get on with algebra as soon as possible. This is why I have kept Chapter 1 to a bare minimum. I didn’t want to include it at all, but the material there is absolutely essential for Chapter 2, and the students’ knowledge of it is apt to be a bit hazy. Other bits of “set theory” will be expounded upon as the need arises. Zorn’s Lemma and some of its applications are discussed in the Appendix.

Groups are chosen as the first algebraic system to study for several reasons. The notion is needed in subsequent topics. The axioms for a group are simpler than those for the other systems to be studied. Such basic notions as homomorphisms and quotient systems appear in their simplest and purest forms. The student can readily recognize many old friends as abstract groups in concrete disguise.

The first topic slated for a thorough study is that of vector spaces. It is this material that is most universally useful, and it is important to present it as soon as is practical. In fact, the arrangement of Chapters 3 through 5 is a compromise between mathematical efficiency and getting the essentials of linear algebra done. Chapter 4 is where it is because its results are beautifully applicable in Chapter 5 and contain theorems one would do anyway. Besides, there are some nice applications of Chapter 3 in Chapter 4. Chapter 6 is basic and should not be slighted in favor of 7 or 8. A feature of Chapter 7 is an algebraic proof of the fundamental theorem of algebra.

There are many exercises in this book. Mathematics is not a spectator sport, and is best learned by doing. The exercises are provided so that students can test their knowledge of the material in the body of the text, practice concocting proofs on their own, and pick up a few additional facts.
There is no need here to extol the virtues of the abstract approach, and
the importance of algebra to the various areas of mathematics. They are well
known to the professional, and will become fully appreciated by the student
only through experience.

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