CANCELLATION IN DIRECT SUMS
OF GROUPS

BY

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Cancellation in Direct Sums of Groups

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1. Introduction. The purpose of this paper is to consider the following question for groups. If \( F \oplus G = F' \oplus H \) and \( F \cong F' \), when is \( G \cong H \)? It is easy to see that for \( G \) to be isomorphic to \( H \) some additional hypothesis must be given. For example let \( C_n \), \( n = 1, 2, 3, \ldots \) be cyclic of order two. Let \( F = C_2 \oplus C_2 \oplus C_2 \oplus \ldots \), let \( G = C_1 \), let \( F' = C_2 \oplus C_2 \oplus C_2 \oplus \ldots \), and let \( H = C_2 \oplus C_2 \oplus C_2 \oplus \ldots \). Clearly \( F \oplus G = F' \oplus H \) and \( F \cong F' \), yet \( G \) not \( \cong H \). One hypothesis that is suggested by this example is to require that \( F \) be finitely generated. In fact, in Kaplansky's book, Infinite Abelian groups, p. 13, the author asks the following question for Abelian groups, called Test Problem III. If \( F \oplus G = F' \oplus H \), \( F \cong F' \), and \( F \) is finitely generated, is \( G \cong H \)? The main results of this paper are theorems from which Test Problem III follows as a corollary.

2. Notations. The additive notation for groups will be used. The symbol \( F \oplus G \) will denote the direct sum of the two groups \( F \) and \( G \). The commutator subgroup of a group \( G \) will be denoted by \( G' \). The symbol \( [S] \) will denote the group generated by the set \( S \) of elements. The order of an element \( g \) of \( G \) will be written \( o(g) \), and \( o(S) \) will denote the number of elements in the set \( S \). The infinite cyclic group will be represented by \( C \), and the cyclic group of order \( n \) by \( C_n \). The symbol \( Z(G) \) will denote the center of the group \( G \).

3. Definitions. Suppose \( F \oplus G = F' \oplus H \). The set of \( F \) components of the elements of \( F' \) will be denoted by \( F \), and the set of \( F \) com-

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4. Theorem. If \( F \oplus G = F' \oplus H \) then \( F + H = F \oplus G = F' \oplus H \).

Proof. It is easy to see that \( F + H = F \oplus (F + H) \cap G \). Let \( g \) be in \( (F + H) \cap G \). Then \( g = f + h \) with \( f \) in \( F \) and \( h \) in \( H \). But \( h = f_1 + g_1 \) with \( f_1 \) in \( F \) and \( g_1 \) in \( G \). Hence \( g = f + f_1 + g_1 \). Since any element can be written in exactly one way as the sum of an element of \( F \) and one of \( G \), it follows that \( f + f_1 = 0 \) and \( g = g_1 \). Thus \( (F + H) \cap G \subseteq G \). Let \( g_2 \) be any element of \( G \). Then \( (g + g_2) \) is in \( H \) for some \( g \) in \( F \). Then \( -(f + g_2) = g_2 \) is in \( G \) so that \( G \subseteq F + H \). Hence \( G \subseteq (F + H) \cap G \), and we get \( F + H = F \oplus G \). Similarly \( F + H = F' \oplus H \).

5. Theorem. If \( F \oplus G = F' \oplus H \), then \( F / (F \cap H) \cong F' \), and \( H / (F \cap H) \cong G \).

Proof. Project \( F \) onto its \( F' \) components. The range of this projection is \( F' \) and the kernel is \( F \cap H \). Hence \( F / (F \cap H) \cong F' \). Similarly \( H / (F \cap H) \cong G \).

6. Theorem. If \( F \oplus G = F' \oplus H \), \( F \cong F' \), and \( F \) is finite then \( G \cong H \).

Proof. By §4, \( F + H = F \oplus G \Rightarrow F' \oplus H \). If \( F \cap H = 0 \), then \( F \oplus H \) exists and \( F \oplus H = F' \oplus H \), so that \( F \cong F' \). Since \( F \) is finite and \( F \cong F' \), it follows that \( F' = F' \) and \( F \oplus H = F' \oplus H = F \oplus G \). Hence \( G \cong H \). If \( F \cap H \neq 0 \), we proceed by induction on \( o(F) = o(F') \). If \( o(F) = 1 \) the theorem is obvious. Since \( F \oplus G = F' \oplus H \), we get that \( F / (F \cap H) \cong F \oplus G \). Since \( F \cong F \), \( F \cong F' \) and \( F \cong F \). By §5 it implies that \( F / (F \cap H) \cong F \cong F' \oplus H \). By §5, \( F / (F \cap H) \cong F' \) and since \( F' \cap H \neq 0 \), it follows that \( o(F') < o(F) \). Hence our induction hypothesis implies that \( G \cong H \).

7. Theorem. If \( F \oplus G = F' \oplus H \), \( F \cong F' \cong C \), and \( G \) is Abelian, then \( G \cong H \).

Proof. Let \( P \) be the projection \( (f'+h)P = f' \), where \( f' \) is in \( F' \) and \( h \) is in \( H \). Then \( P \) is a homomorphism of \( G \) into \( F' \) with kernel \( G \cap H \). If \( GP = 0 \), then \( G \subseteq H \) and since \( G \) is a direct summand of a group containing \( H \), we get \( H = G \oplus H' \) for some group \( H' \). Hence \( F \oplus G = F' \oplus G \oplus H' \) and \( F \cong F' \oplus H' \). Since \( F \cong F \), \( F \cong F'H' \) and since \( H' = 0 \) and \( G \cong H \). If \( GP \neq 0 \) then \( GP \) is infinite cyclic, so that \( G / (G \cap H) \cong C \), from which follows that \( G \cong (G \cap H) \oplus C \). Since \( G \)

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1 Bjørn H. Jónsson has informed me that this result follows from Theorem 3.11 in the book "Direct Decomposition of Finite Algebraic Systems," by Jónsson and Tarski.
Abelian. The group $H$ is also Abelian so by symmetry we may assume $H \cong (H \cap G) \oplus C$. Hence $G \cong H$.

8. Corollary (Kaplansky's Test Problem III). If $F \oplus G = F' \oplus H$, $F \cong F'$ and $F$ is finitely generated, and $F$ and $G$ are Abelian, then $G \cong H$.

Proof. Since $F$ is Abelian and finitely generated it is the direct sum of a finite group and a finite number of infinite cyclic groups. Clearly it suffices to prove $G \cong H$ for the two cases $F$ finite and $F$ infinite cyclic. But these two cases are proved in §§6 and 7. Hence §8 follows.

9. Corollary. The decomposition of a finitely generated Abelian group into indecomposable summands is unique up to isomorphism.

Proof. Let $F \oplus F_1 \oplus \cdots \oplus F_n = F' \oplus F'_1 \oplus \cdots \oplus F'_{n'}$ be two decompositions of a finitely generated Abelian group into indecomposable summands. Then each $F_i$ and $F'_i$ is cyclic of prime power order or infinite cyclic. We induct on $m$. If $m = 1$, then §9 follows because $F_1$ and $F'_1$ are indecomposable. Suppose the left side has an infinite cyclic summand. Then $m$ does the right. Let for convenience these two summands be $F_1$ and $F'_1$. Then by §8, $F_1 \oplus F_2 \oplus \cdots \oplus F_n \cong F'_1 \oplus F'_2 \oplus \cdots \oplus F'_{n'}$ and by the induction hypothesis §9 follows.

Now we may assume all the summands on the left side are finite.

Let $F_i$ be a summand of maximal prime power order, say $p^r$. Hence $F_t$ has an element $f_t$ of order $p^r$. Furthermore $f_t = f_t + f_t' + \cdots + f_t''$ and some $f_t''$ is of order $p^r$. If $o(f_t) < o(F_t)$ then the right side has an element of order $p^r$ with $r < c$ and the left does not. Hence $o(F'_t) = o(f_t)$ and by §8 and the induction hypothesis we are done.

10. Theorem. If $F \oplus G = F' \oplus H$, then the following are true:
   (a) $Q(F_t) \subseteq F \cap F'$.
   (b) $Q(F_t) \subseteq F \cap H$.
   (c) $Q(F) = Q(F_t) \oplus Q(F_t)$.
   (d) $Q(F) = Q(F'_t) \oplus Q(H_t)$.

Proof. The proof is straightforward and routine, so is omitted.

11. Theorem. If $F \oplus G = F' \oplus H$, $F \cong F' \cong C$, then the following are true:
   (a) $Q(G) = Q(H)$.
   (b) $Q(G) \cong H / Q(f_t)$.
   (c) The group $G$ is isomorphic to a normal subgroup of $H$ of finite index in $H$. In fact $G \cong H / Q(f_t)$.
   (d) The group $H$ is isomorphic to a normal subgroup of $G$ of finite index in $G$. In fact $H \cong G$ or $H \cong G$. 

(c) If \( Z(G) \) is torsion then \( G \cong H \). (In particular, if \( G \) is finite or if \( Z(G) = 0 \), then \( G \cong H \).)

(l) If \( Q(G) = G \) then \( G \cong H \).

Proof. (a) Using \( \# 10 \) applied to \( G \) and \( H \) and the fact that \( F \) and \( F' \) are Abelian, we get that \( Q(G) = Q(G_2) \) and \( Q(H) = Q(H_2) \). But \( Q(G) = Q(F_1) \oplus Q(H_1) = Q(H_2) \) implies \( Q(G) = Q(H) \).

(b) Since \( Q(G) = Q(H) \) we get \( F \oplus G \cong Q(G) \cong F' \oplus H \cong Q(H) \). The groups \( G/Q(G) \) and \( H/Q(H) \) are Abelian. By \( \alpha \), \( G/Q(G) \cong H/Q(H) \).

(c)-(d) Let \( F = \{ f \} \) and \( F' = \{ f' + g \} \). Then \( f = (mnf + mg) + ((1 - mm)f - mg) \), with \( mnf + mg \) in \( F' \) and \( ((1 - mm)f - mg) \) in \( H \).

If \( m = 0 \) then \( F \subseteq G \). In that case \( G = F \oplus G' \), for some \( G' \). Thus \( F \oplus F' \oplus G = F' \oplus H \) and so \( H \cong F \oplus G' \cong F' \oplus G' \). Hence \( G \cong H \). If \( m = 1 \) then \( F' \cap H = 0 \) and the projection of \( H \) onto \( G \) is an isomorphism. Since \( \{ G_1, g \} = G \), and \( mg \) is in \( G_1 \), it follows that the index of \( G_1 \) in \( G \) is infinite. If \( mm \neq 1 \) and \( mn \neq 0 \), then \( F' \cap H = 0 \) because \( a(f) \) is infinite and so the \( F' \) component of \( F \) is zero.

(c) and (d) are proved.

In proving (c) and (d) it was shown that either \( G \cong H \) or \( F \cap H = 0 \). If \( F' \cap H = 0 \) then \( ((1 - mm)f - mg) \) is the \( H \) component of \( f \); it follows that \( \sigma(g) \) is infinite. But \( F' = \{ nf + g \} \) implies that \( g \) is in \( Z(G) \). Thus \( Z(G) \) is torsion, \( F' \cap H = 0 \) and \( G \cong H \).

If \( Q(G) = G \), then by (b), \( Q(H) = H \). Using (a), we get \( G = H \).

12. Remark. Professor W. R. Scott constructed the interesting example given in the next theorem.

13. Theorem. There exist groups \( G \) and \( H \) such that \( G \not\cong H \), yet \( F \cong G = F' \oplus H \) with \( F \cong F' \cong C \).

Proof. Let \( G \cong H = \{ x \} \) with \( 11x = 0 \). Let \( G \) be the extension of \( G' \cong H \) by the automorphism \( A \) such that \( xA = 2x \). That is, \( G = \{ x, y \} \) where \( -y + x + y = 2x \) and \( G/(G' \cong H) \cong \{ y \} \cong C \). Let \( H \) be the extension of \( G' \cong H \) by the automorphism \( A \) such that \( xA = 8x \). That is \( H = \{ x, z \} \) where \( -z + x + z = 8x \) and \( H/(G' \cong H) \cong \{ x \} \cong C \). Now let \( F = \{ y, z \} \), \( F' = \{ y, 3z \} \), and \( 2y + y = y + z \). It is straightforward to prove that the groups just defined have the desired properties.

14. Theorem. If \( F \cong G = F' \oplus H \), \( F' \cong C \), then \( G \cong H \).

Proof. By \( \# 10 (c) \), \( Q(F) = Q(F_1) \oplus Q(F_2) \). Since \( F \) is indecomposable, \( Q(F) = Q(F_1) \) and \( Q(F_2) = 0 \), or \( Q(F) = Q(F_2) \) and \( Q(F_1) = 0 \). In the first
case $F \leq F'$ and by §10(4), $F = Q(F) = Q(F')$. Since $F \geq F'$ is indecomposable and $F' = Q(F') = Q(F') \oplus Q(F')$ it follows that $F = F'$ and that $G \cong H$. Similarly if $Q(F) = Q(F')$ and $Q(F_0) = 0$ then $G \cong H$.

15. Corollary. If $F \oplus G = F' \oplus H$, $F \cong F'$ and $F$ is simple, then $G \cong H$.

Proof. If $F$ is non-Abelian, §15 follows from §14. If $F$ is Abelian §15 follows from §6.

16. Concluding remarks. The theorems given in §§4, 5, and 6 can be generalized to operator groups. In §6, instead of requiring in the operator group case that $F$ be finite, one requires that $F$ have only a finite number of admissible subgroups. Kaplansky’s Test Problem III may be extended to modules over principal ideal rings, and if $F$ is torsion, to modules over Dedekind rings. The proof given in §9 of the uniqueness of the decomposition of finitely generated Abelian groups is, in this writer’s opinion, a much better proof than has existed here-tofore.

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