Cotorsion Free, an Example of Relative Injectivity

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1. Introduction

If $\mathcal{P}$ is a class of abelian groups we may consider the class $\mathcal{E}$ of short exact sequences $0 \to A \to B \to C \to 0$ such that $0 \to \text{Hom}(P, A) \to \text{Hom}(P, B) \to \text{Hom}(P, C) \to 0$ is exact for all $P \in \mathcal{P}$. The class is the class of proper short exact sequences determined by declaring the groups in the class $\mathcal{P}$ to be projective. The maps $A \to B$ and $B \to C$ are referred to as proper monics and proper epics. The class $\mathcal{F}$ of all groups $F$ such that $0 \to \text{Hom}(C, F) \to \text{Hom}(B, F) \to \text{Hom}(A, F) \to 0$ is exact for all sequences $0 \to A \to B \to C \to 0$ in $\mathcal{E}$ are the relative injectives. The class of relative projectives, defined dually, includes $\mathcal{P}$, generally properly.

If $\mathcal{P}$ is taken to be the class of torsion groups the reduced elements of $\mathcal{F}$ are exactly the (reduced) cotorsion groups. This and general duality terminology motivate the following definition. If $T$ is a term describing the groups of the class $\mathcal{F}$ then the reduced groups of the class $\mathcal{F}$ are called co $T$ groups. Similarly, groups which have no nonzero $T$ subgroups are called $T$ free. The purpose of this note is to show that among reduced groups, the class of cotorsion free groups is coextensive with the class of co torsion free groups.

Cotorsion groups were introduced by HARRISON in [2]. The basic facts we use about them either appear there or belong to folklore. For a treatment of relative homological algebra see [1].

2. The Associative Law

We deal throughout with the relative homological algebra induced in the category of abelian groups by declaring torsion free groups to be projective. All terms such as "proper" and "injective" refer to this setting.

**Proposition 1.** If $A$ is cotorsion then any exact sequence $0 \to A \to B \to C \to 0$ is proper.

**Proof.** Consider the exact sequence $\text{Hom}(F, B) \to \text{Hom}(F, C) \to \text{Ext}(F, A)$ for torsion free $F$. Since $A$ is cotorsion $\text{Ext}(F, A) = 0$ and so $0 \to \text{Hom}(F, A) \to \text{Hom}(F, B) \to \text{Hom}(F, C) \to 0$ is exact which shows that $0 \to A \to B \to C \to 0$ is proper.

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Proposition 2. Every reduced injective is cotorsion free.

Proof. Let \( G \) be a reduced injective and \( K \) a nonzero cotorsion subgroup of \( G \). Then by Proposition 1 the inclusion map \( K \to G \) extends to the divisible hull of \( K \) which contradicts the fact that \( G \) is reduced.

Proposition 3. For every group \( C \) there is an exact sequence \( 0 \to A \to B \to C \to 0 \) with \( B \) torsion free and \( A \) cotorsion.

Proof. Let \( \mathcal{C} \) be the class of groups \( C \) for which the proposition holds. Then \( \mathcal{C} \) is closed under the taking of subgroups and finite direct sums. Since every group is a subgroup of the direct sum of a divisible torsion and a torsion free divisible, and since all torsion free groups are in \( \mathcal{C} \), we need only show that every divisible torsion group is in \( \mathcal{C} \). If \( D \) is a divisible torsion group let \( \Lambda \) be a product of copies of the \( p \)-adic integers, at least as many copies for each prime \( p \) as the rank of \( D \). If \( K^* \) is the divisible hull of \( K \) then \( K^*/K \) contains a copy of \( D \) and is in \( \mathcal{C} \).

Corollary 1. There are enough projectives, i.e. every group is the proper image of a projective.

Corollary 2. Every projective is torsion free.

Proposition 4. If \( A \) is cotorsion free, \( B \) is torsion free and \( A \subseteq B \) is proper then \( A \) is a summand of \( B \).

Proof. If \( \mathbb{B}/A \) is torsion free we are done. Otherwise \( \mathbb{B}/A \) contains the \( p \)-element group \( Z_p \) for some prime \( p \). This yields a proper exact sequence \( 0 \to A \to C \to Z_p \to 0 \) where \( C \subseteq B \). Since the \( p \)-adic integers \( \mathbb{P} \) are torsion free the natural map \( P \to Z_p \) lifts to a map \( P \to C \). Since \( P \) is cotorsion the image \( C \) of this map is the direct sum of a cotorsion group and a divisible group. Since \( Q \) maps onto \( Z_p \) it cannot be divisible. But \( pQ \subseteq Q \) since \( Q \) is a subgroup of the torsion free group \( C \). Also \( pQ \subseteq A \) since \( pC \subseteq A \), but this contradicts the assumption that \( A \) is cotorsion free.

We are now in a position to prove an associative law.

Theorem. If \( G \) is a reduced group then \( G \) is cotorsion free if and only if \( G \) is a torsion free.

Proof. \( \Rightarrow \) Proposition 2.
\( \Leftarrow \) Let \( G \) be reduced cotorsion free, \( 0 \to A \to B \to C \to 0 \) proper and \( A \to G \). We must extend \( f \) to \( B \). Since the sequence \( 0 \to A/\ker f \to B/\ker f \to C \to 0 \) is proper we may assume that \( f \) is monic, or, in fact, that \( A \to G \). The problem is thus reduced to showing that if \( A \) is cotorsion free and \( A \subseteq B \) is proper that \( A \) is a summand of \( B \).

Let \( F \to B \) be an epic with cotorsion kernel \( K \) and \( F \) torsion free (Proposition 3). If \( \phi \) is the preimage of \( A \) under this map we have the diagram

\[
\begin{array}{ccc}
\phi & \to & F \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}
\]
with proper epic columns and proper monic row. Since $A$ is cotorsion free and therefore torsion free the left hand map splits and we can write $\Phi = A \oplus K$. Thus $A \oplus F$ is proper and so splits and $F = A \oplus L$. Now if $L \cong K$ then the projection of $F$ on $A$ gives a nonzero cotorsion subgroup of $A$, a contradiction. Thus the projection of $F$ onto $A$ factors through $B$ and $A$ is a summand of $B$.

3. The Purity of Injectives

Although the associative law suggests a certain symmetry in this setting it turns out that although there are sufficiently many projectives, there are not enough injectives, a contrast to the torsion-cotorsion set up.

**Proposition 5.** Let $G$ be an unbounded reduced $p$-group. Then $G$ is not a proper subgroup of an injective.

**Proof.** Suppose $G$ is a proper subgroup of an injective $I$. By the theorem, we may assume $I$ is divisible. Then $\text{Ext}(Q, G) \cong \text{Ext}(Q, I) = 0$ where $Q$ is the additive group of rational numbers. But this implies that $G$ is cotorsion and hence bounded, a contradiction.

**References**


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