Direct-Sum Representations of Injective Modules

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One of the main results of this article is the following:

A. Theorem 5.3. The following conditions on a ring $R$ are equivalent:
   (1) $R$ is quasi-frobenius; (2) each injective right $R$-module is projective; (3) each injective left $R$-module is projective.

   The "dual" theorem obtained by the substitutions "injective" $\Rightarrow$ "projective" in this statement is the subject of another paper [10]. If $(2^*)$ [resp. $(3^*)$] denotes the statement dual to (2) [resp. (3)], we thus obtain the equivalence of the statements: (1), (2), (3), $(2^*)$, and $(3^*)$.

   The observation that (2) implies that each injective module is contained in a direct sum of cyclic $R$-modules led to the more general study indicated by the title. Our first theorem of this type is as follows:

B. Theorem 1.1 and 3.3. The following conditions on a ring $R$ are equivalent:
   (1) $R$ is right noetherian.
   (2) There exists a cardinal number $c$ such that each injective right $R$-module is a direct sum of modules generated by $c$ elements.
   (3) There exists a cardinal number $d$ such that each injective right $R$-module is contained in a direct sum of modules generated by $d$ elements.

   A corollary is a theorem of Papp [19]: If $R$ is a ring such that each injective right $R$-module is a direct sum of indecomposable modules, then $R$ is right noetherian.

We also obtain:

C. Theorem 3.1. If each right $R$-module is contained in a direct sum of finitely generated modules, then $R$ is right artinian.

This generalizes the theorem of Cohen and Kaplansky [6] (for commutative rings) and Chase [5] (for non-commutative rings) which states that $R$ is

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right artinian in case each right $R$-module is a direct sum of finitely generated modules. (The Cohen-Kaplansky theorem was further restricted to direct sums of cyclic modules.) Combining C with a theorem of Mocia, we obtain:

**D. COROLLARY 3.2.** A commutative ring $R$ is artinian if and only if each injective right $R$-module is a direct sum of finitely generated modules.

If $R$ is a ring with property of $C$, then it is right noetherian by $B$, and the following result is applied to deduce that $R$ is right artinian.

**E. COROLLARY 2.3.** If $R$ is right noetherian, and if the injective hull of each cyclic (resp. finitely generated) right $R$-module is finitely generated, then $R$ is right artinian.

The proof of $E$ makes use of Goldie's theorem. We divide out the maximal nilpotent ideal $X$ and $R = R/N$ inherits the hypothesis, namely, the injective hull of $R$ as an $R$-module is finitely generated. But $R$, being semi-prime, has a classical right quotient ring $Q$ which is a semi-simple ring. Furthermore, $Q$ is isomorphic to the injective hull of $R$, considered as an $R$-module. So in the case of quartet fields of integral domains, $Q$ is a noetherian $R$-module only if $Q = R$. Thus, $R = R/N$ is semi-simple, so, in the presence of the a.c.c. on right ideals, $R$ is right artinian.

Going back to $A$, if each injective module is projective, then each module is contained in a direct sum of copies of $R$, and hence $R$ is a cogenerator in the category $M_R$ of right $R$-modules (see Section 4). We prove the following theorem to deduce that $R$ is right self-injective: (R is right artinian by $C$).

**F. THEOREM 4.1.** If $M_R$ has a finitely generated projective cogenerator $P$, and if $R$ islo $R$ is semi-simple, then $P$ and $R$ are injective in $M_R$.

Thus (2) of $A$, implies that $R$ is right self-injective and right artinian, so $R$ is quasi-frobenius by a theorem of Hida [23] (see Section 5).

Among other things we prove that any indecomposable projective and injective module over $R$ is a direct summand of $R$ (Corollary 2.5). Section 6 is a study of direct summands of completely decomposable modules. The principal result there is that if $Q = \bigoplus Q_i$ is a direct sum of indecomposable injective modules, then any direct summand of $Q$ is a direct sum of some of the $Q_i$'s, provided either $Q$ is injective, or the $Q_i$ are countably generated.

**G. BACKGROUND**

Throughout this paper each ring $R$ will be a ring with identity element $1$, and each module $M$ will be unitary in the sense that $x1 = x$ for all $x \in M$. $\mathfrak{N}_R$ (resp. $\mathfrak{N}_L$) will denote the category of all right (resp. left) $R$-modules.
If $M \in \mathfrak{M}$ then variously $\hat{M}$, or inj. hulla$M(M)$, will denote the injective hull of $M$ (Eckmann-Schepf [7]).

In any category, direct products of injective objects are injective. However, direct sums of injective objects are not in general injective. We shall have occasion to use the following theorem (Cartan-Eilenberg-Bass-Papp) several times in the sequel.

0.1. Theorem. The following conditions on a ring $R$ are equivalent:

(a) $R$ is right noetherian,
(b) Any direct sum of injective right $R$-modules is injective.
(c) Any countable direct sum of injective right $R$-modules is injective.

Proof. (a) $\Rightarrow$ (b) is an exercise in Cartan-Eilenberg [4]. (b) $\Rightarrow$ (c) is trivial, while (c) $\Rightarrow$ (a) is Bass's observation (see Chase [5], p. 471 and Papp [79]).

An injective right $R$-module $M$ is said to be $\Sigma$-injective in case a direct sum of arbitrarily many copies of $M$ is injective; $M$ is countably $\Sigma$-injective in case a direct sum of $\aleph$-many copies of $M$ is injective. In Section 6 we use the next theorem which was proved in [10].

0.2. Theorem. An injective module $M$ is $\Sigma$-injective if and only if it is countably $\Sigma$-injective.

Proof. We give a short proof which replaces the laborious proof given in [10].

Let $M$ be countably $\Sigma$-injective, and let $Q = \sum_{n \in \mathbb{N}} Q_n$, be a direct sum of arbitrarily many copies $Q_n$ of $M$. For any subset $A$ of $I$, let $Q_A = \sum_{n \in A} Q_n$, and let $\pi_A$ denote the projection of $Q$ on $Q_A$ having kernel $Q_{I-A}$. In case $A = \{a\}$, denote $\pi_a$ by $\pi_a$.

Next, assume $Q$ is not injective. Then there exists a right ideal $S$ of $R$, and a map $f : S \to Q$ which is not extendable to a map $g : R \to Q$. In particular, there does not exist a finite subset $F$ of $I$ such that $f(S) \subseteq Q_F$ (since $Q_F$ is injective and a direct summand of $Q$). This implies that there exist a countably infinite subset $A$ of $I$ such that $\pi_a f \neq 0 \forall a \in A$. Then $\pi_a f = \pi_a f \neq 0$ for any $a \in A$, so $\pi_a f(S)$ is not contained in $Q_B$ for any finite subset $B$ of $A$. Since $Q_A$ is injective by hypothesis, there exists a mapping $h : R \to Q_A$ extending $\pi_a f : S \to Q_A$. But $h(R)$, being a cyclic submodule of $Q_A$, is contained in $Q_B$ for some finite subset $B$ of $A$. Then $\pi_B f(S) \subseteq h(R) \subseteq Q_B$, contradicting the assertion above, and proving that $Q$ must be injective.

Although we will not use the following theorem, we give it because of its connection with Theorem 0.2.

Theorem. If each injective module in $\mathfrak{M}$ is countably $\Sigma$-injective, then
R is right noetherian (and hence arbitrary direct sums of injective modules in \( \mathfrak{M}_R \) are injective).

**Proof.** By Theorem 0.1, it suffices to show that any direct sum \( Q = \sum_{i=1}^\infty Q_i \) of countably many injective modules \( \{Q_i | i = 1, 2, \ldots \} \) is injective. Now \( M = \prod_{j=1}^\infty Q_j \) is injective, and, by hypothesis, a direct sum \( P = \sum_{i=1}^\infty M_i \) of countably many copies \( \{M_i | i = 1, 2, \ldots \} \) of \( M \) is injective. But \( M_i \) contains as a direct summand a copy \( Q_j \) of \( Q_j \), \( j = 1, 2, \ldots \cdot \). Write \( M_i = Q_j \oplus P_i \). Then \( P = \sum_{i=1}^\infty Q_j \oplus \sum_{i=1}^\infty P_i \), so that \( Q_j \), being isomorphic to the direct summand \( \sum_{i=1}^\infty Q_j \oplus P_i \) of \( P \), is injective.

In Section I we shall use the following theorem of Matlis [16] and Papp [19].

0.3. **Theorem.** If \( R \) is right noetherian, then any injective right \( R \)-module is a direct sum of indecomposable modules.

Z. Papp has shown that this property characterizes noetherian rings (cf. Corollary 1.3).

1. **Characterizations of Noetherian Rings**

1.1. **Theorem.** \( R \) is right noetherian if and only if there exists a cardinal number \( c \) such that each injective right \( R \)-module is a direct sum of modules each generated by \( c \) elements.

**Proof.** If \( R \) is right noetherian, then by Theorem 0.3, each injective \( M \in \mathfrak{M}_R \) is a direct sum of indecomposable injective modules. Since an indecomposable injective module \( D \) is the injective hull \( C \) of any nonzero cyclic submodule \( C \), it suffices to show that there exists a cardinal number \( c \) such that each such \( D \) is generated by \( c \) elements. Since the collection of all isomorphism classes of cyclic modules is a set, it follows that the collection of all isomorphism classes of indecomposable injective modules is a set \( \{\mathfrak{M}_i | i \in I\} \). If \( M \in \mathfrak{M}_i \) is generated by \( c_i \) elements, then \( c = \sum_{i \in I} c_i \) (cardinal sum) is the desired cardinal.

Conversely, assume that such a cardinal number exists. \( R \) is right noetherian if and only if each direct sum of injective modules is injective (Theorem 0.1). By our assumption it suffices to show that if \( M \) is a direct sum \( \sum_{i=1}^\infty M_i \) of injective modules \( M_i \), each generated by \( c_i \) elements, then \( M \) is injective. For simplicity let \( c \) be an infinite cardinal \( \geq | R | \). We may assume that \( I \) is infinite.

Let \( B \) be a set with cardinality \( > 2^c \), where \( d = | I | \). For each \( i \in I \), let \( N_i = \prod_{b \in B} M_i, b \), the direct product of \( | B | \) copies of \( M_i \), and let \( P = \prod_{i \in I} N_i \).
Since a direct product of injective modules is always injective, $N_i$ is injective for all $i$, and $P$ is injective. By hypothesis, we may write $P = \sum_{i \in I} Q_i$, where $Q_i$ is generated by $c$ elements. Well-order $I$, and take one of the direct summands $M_{i,k}$ of $N_i$. Since $M_{i,k}$ is generated by $c$ elements, and since $c$ is infinite, and since each element of $M_{i,k}$ is contained in a direct sum of finitely many $Q_i \mid g_i \in G$, then $M_{i,k}$ is contained in $P = \sum_{i \in I} Q_i$, where $G_i$ is a subset of $G$ consisting of $c$ elements. Since each $Q_i$ is generated by $c$ elements, $P_i$ is generated by $c^2$ elements. Consequently $|P_i| \leq c^2 \leq |R| \leq c^2 - c$, so $P_i$ has at most $2^c$ subsets. Since $(M_{i,k} \cap P_i \mid b \in B)$ is an independent collection of submodules of $P_i$, and since $|B| > 2^c$, then $M_{i,k} \cap P_i = 0$, for some $b \in B$. The projection $\delta$ of $M_{i,k}$ into $\sum_{\text{all } Q_i}$ is a monomorphism, and $\delta(M_{i,k}) \subseteq P_i = \sum_{i \in I} Q_i$, where $G_i$ is a subset of $G$ consisting of $c$ elements, and $G_i \cap G_j$ is empty.

For $x \in I$, assume that there exist mutually disjoint subsets $(G_{k,x})_{k \in I}$ of $G$ such that $G_k$ contains $c$ elements, and such that $P_x = \sum_{i \in I} Q_i$ contains a copy $M_{i,k}$ of $M_x$. Let $H_x = \bigcup_{k \in I} G_k$, and set $S_x = \sum_{i \in I} Q_i = \sum_{i \in I} P_i$. Since each $P_i$ is generated by $c$ elements, and since $|H_x| \leq c \cdot d$, it follows that $|S_x| \leq c \cdot d \cdot c = c \cdot d^2$, so that $S_x$ has at most $2^{c \cdot d^2}$ subsets. Since $|B| > 2^{c \cdot d^2}$, by the reasoning above, there exists a $b \in B$ such that $M_{i,k} \cap S_x = 0$. Thus there exists a subset $G_{k,x}$ disjoint from $H_x$ such that $G_k$ is generated by $c$ elements, and such that $P_x = \sum_{i \in I} Q_i$ contains a copy $M_{i,k}$ of $M_x$. By transfinite induction $G_k$ and $P_x$ exist $\forall x \in I$. Let $H = \bigcup_{x \in I} G_k$. Then

$$P = \sum_{i \in I} P_i \oplus \sum_{x \in I} Q_x.$$ 

Since $M_{i,k}$ is injective, and since $P_x$ contains an isomorphic copy, $M_{i,k}$ is isomorphic to a direct summand of $P_x$, $\forall x \in I$. Since $\sum_{i \in I} P_i$ is a direct summand of $P$, and since $P$ is injective, it follows that $\sum_{i \in I} M_{i,k}$, being isomorphic to a direct summand of $P$, is injective. Since $\sum_{i \in I} M_{i,k}$, $M$ is injective.

The condition stated in the theorem seems first studied by S.U. Chase [5, 471, Lemma 4.1]. There, in a lemma contributed by Bass, it is proved that any semi-primary ring $R$ satisfying it is a right Artinian. $(R$ is semi-primary in case $R/\text{rad } R$ is semisimple and $R/\text{nilpotent}$.

1.2. COROLLARY. A ring $R$ is right noetherian if and only if there exists a cardinal number $d$ such that every injective module in $\mathcal{M}_R$ is a direct sum of injective tails of modules generated by $d$ elements.

Proof. The necessary follows from the theorem. Let $F$ denote the free module with a free basis of cardinality $d$. Then each element of $F$ is a
direct sum of injective hulls of modules of the form $F/K$. Since $\{F | K \text{ a submodule of } F\}$ is a set, $\{F/K | K \text{ a submodule of } F\}$ is a set. If $F/K$ is generated by $e_K$ elements, then each module in $\{F/K | K \text{ a submodule of } F\}$ is generated by $e = \sum e_K$ elements. Thus, there exists a cardinal $e$ with the property stated in the theorem, so the theorem applies. Thus $R$ is right noetherian.

1.3 COROLLARY (Theorem of Papp [19]). If each injective module in $\mathbb{M}_R$ is a direct sum of indecomposable modules then $R$ is right noetherian.

Proof. If $N$ is an indecomposable injective module, then $N$ is the injective hull of any nonzero cyclic module, so $R$ is right noetherian by the corollary.

Modifying Bass's terminology slightly, we say that a collection of modules is an injective decomposition basis for $\mathbb{M}_R$ in case each injective module in $\mathbb{M}_R$ is a direct sum of injective hulls of modules in that collection. In this language, the corollary becomes

1.4. COROLLARY. A ring $R$ is right noetherian if and only if there exists a cardinal number $d$ such that the modules in $\mathbb{M}_R$ generated by $d$ elements form an injective decomposition basis for $\mathbb{M}_R$.

There is a similar characterization of artinian rings.

1.5. COROLLARY. A ring $R$ is right artinian if and only if the semi-simple modules in $\mathbb{M}_R$ form an injective decomposition basis for $\mathbb{M}_R$.

Proof. If $R$ is right artinian, then $R$ is right noetherian, so each injective $M \in \mathbb{M}_R$ is a direct sum of indecomposable injective modules. If $M$ is indecomposable injective, then $M$ is the injective hull of any nonzero submodule. Since $R$ is right artinian, each nonzero module contains a simple submodule. Consequently, each indecomposable injective is the injective hull of a simple module, proving one part.

The converse depends on theorems of H. Bass, C. Hopkins, and J. Levitzki. A ring $R$ is left perfect in case each left $R$-module has a projective cover. We do not use this concept, but rather the equivalence of the other conditions in:

BASS'S THEOREM [3]. The following conditions on a ring $R$ are equivalent:

(1) $R$ is left perfect.

(2) $R/\text{rad } R$ is semi-simple, and $\text{rad } R$ is right T-nilpotent.

An equivalent formulation is: $R$ is right noetherian if and only if some injective decomposition basis of $\mathbb{M}_R$ forms a set.
(3) \( R \) satisfies the d.c.e. on principal right ideals.
(4) \( R \) has no infinite set of orthogonal idempotents, and each nonzero right \( R \)-module has nonzero socle.

C. Hopkins [12] and J. Levitzki [14] proved that any right artinian ring is right noetherian. Hopkins' proof has the following corollary, which we need:

A semi-primary ring \( R \) is right noetherian if and only if it is right artinian.

**Proof of the converse.** If \( M_R \) has such an injective decomposition basis, then \( R \) is right noetherian by the corollary. Hence, by the result of C. Hopkins, it suffices to prove that \( R \) is semi-primary.

Let \( M \) be any nonzero module, and \( \overline{M} \) its injective hull. Since socle \( (\overline{M}) \neq 0 \), then socle \( M \neq 0 \). Since \( R \) is right noetherian, any collection of orthogonal idempotents in \( R \) is necessarily finite. Since (4) of Buss's theorem is satisfied, we conclude (2): \( \text{rad} R \) is semi-simple, and \( \text{rad} R \) is right \( T \)-nilpotent. In particular, \( \text{rad} R \) is nil. But a theorem of Levitzki [15] states that in a right noetherian ring, any nil ideal is nilpotent. Consequently, \( R \) is semi-primary as needed.

### 2. INJECTIVE HULLS OF FINITELY GENERATED MODULES

In the next section we encounter the following condition: If \( C \) is a cyclic or finitely generated module, then \( \overline{C} \) is finitely generated. This condition for artinian rings was studied by Rosenberg and Zelinsky in [2]. We show that any noetherian ring satisfying this condition must be artinian. Characteristically, a single cyclic module, namely \( R \) modulo the maximal nilpotent ideal \( N \), does the damage.

If \( M, N \in \mathcal{M}_R \), and \( M \supseteq N \), then \( M \cap N \), or \( (M \cap N)_R \), signifies that \( M \) is an essential extension of \( N \) (Fekkenn-Schoof [7]). A right noetherian semi-prime ring is just a right noetherian ring with no nilpotent ideals.

#### 2.1. Lemma. If \( R \) is a right noetherian semi-prime ring, and if \( \hat{\mathcal{M}} = \text{inj. hull} \ \mathcal{M} \) is finitely generated in \( \mathcal{M}_R \), then \( R \) is semi-simple, and \( R = \bar{R} \).

**Proof.** By Coldie's theorem [11], \( R \) is a unique classical right quotient ring \( Q = \{ab^{-1} | a, \text{regular } b \in R \} \) which is a semi-simple ring. If \( q = ab^{-1} \) and if \( q \neq 0 \), then \( a = qb \) is a nonzero element of \( q \mathcal{M} \cap R \), showing that \((Q \cap R)_R\). Thus, the natural right \( R \)-module \( Q \) is contained in \( R \). It is not hard to show that \( Q = \bar{R} \), and that the ring operation in \( Q \) induces the module operation in \( R \) (see [9]).
Since $\mathcal{R}$ is finitely generated, it is noetherian. If $b$ is a regular element of $R$, then $b^{-1} \in \mathcal{O}$, and $b^{-n}a = b^{-(n+1)}(ba), \forall a \in R$, showing that
\[ R \subseteq b^{-1}R \subseteq \cdots \subseteq b^{-n}R \subseteq \cdots. \]
Since $\mathcal{R}$ is noetherian, $b^{-n}R = b^{-(n+1)}R$ for some $n$, so $b^{-(n+1)} = b^{-n}a$, for some $a \in R$. But then, $b^{-1} = a \in R$. Since this is true for all regular $b \in R$, it follows that $Q = R$, that is, $R$ is semi-simple.

2.2. Theorem. If $R$ is a right noetherian ring, and $N$ is its maximal nilpotent ideal, and if $R/N = \text{inj. hull } \mathcal{R}(R/N)$ is finitely generated in $\mathcal{M}_R$, then $R$ is right artinian.

Proof. Let $Q$ be the injective hull of $R/N$ in $\mathcal{M}_{R/N}$. Then $Q$ is an $R$-module and as such is an essential extension of $R/N$. Thus we may assume the $R$-module inclusions $R/N \subseteq Q \subseteq R/N$. Since $R/N$ is finitely generated and $R$ is noetherian, $Q$ is finitely generated as an $R$-module and hence as an $R/N$-module. By the lemma, $R/N$ is semi-simple. This implies that $R$ is a semiprimary right noetherian ring, so the theorem of Hopkins stated in Section 1 yields $R$ right artinian.

2.3. Corollary. If $R$ is right noetherian, and if the injective hulls of cyclic (resp. finitely generated) modules in $\mathcal{M}_R$ are finitely generated, then $R$ is right artinian.

2.4. Proposition. Let $a \geq b$ be cardinals with $a > b$. Suppose $C \in \mathcal{M}_R$ is generated by $b$ elements and $\hat{C}$ is contained in a direct sum of modules each of which is generated by fewer than $a$ elements. Then

(i) if $a = \chi_b$, then $\hat{C}$ is finitely generated;
(ii) if $b \geq \chi_a$, then $\hat{C}$ is generated by a family.

Proof. Let $\{Q_i : i \in I\}$ be modules in $\mathcal{M}_R$ each generated by $a$ elements such that $\hat{C}$ is contained in their direct sum. Since each generator of $C$ is contained in a direct sum of finitely many of the $Q_i$, it follows that $C \in \mathcal{M}_R$ contained in $K = \sum_{i \in I} Q_i$, where $I$ is a subset of $I$ with the properties:
\[ b < \chi_b \Rightarrow c = \text{card } I < \chi_b \]
\[ b \geq \chi_a \Rightarrow c = \text{card } I = b. \]

Now let $f$ denote the projection of $\sum_{i \in I} Q_i$ onto $K$. Since $\text{Ker}(f) \cap C = 0$, then $\text{Ker}(f) \cap C = 0$, showing that $f$ maps $C$ monomorphically into $K$. If $\beta = \chi_b$, then $\eta < \chi_b$ and $t < \chi_b$, so $K$ is a direct sum of finitely many finitely generated modules, that is, $K$ is finitely generated. But then, so is any direct summand of $K$; in particular $f(C)$, whence $C$ is finitely generated.
If $b \geq \chi_0$, then $c = b$, so $K$ is generated by $ba = c$ elements, so $f(C)$ and $\mathcal{C}$ are each generated by a elements in this case.

It is known that if $R$ is a QF-ring, then any indecomposable injective module is isomorphic to a direct summand of $R$. Since each injective module over a QF-ring is projective (see Section 5), this is a special case of the following.

2.5. Corollary. If $R$ is any ring, then an indecomposable injective and projective right $R$-module $M$ is isomorphic to a direct summand of $R$; that is, there exists an idempotent $e \in R$ such that $M = eR$.

Proof. Write $M = \mathcal{C}$, where $C$ is any nonzero cyclic submodule of $M$. Since $M$ is projective, $\mathcal{C}$ is contained in a direct sum of copies of $R$, and the proof of the proposition shows that $\mathcal{C}$ is contained in a direct sum $R^{(a)} = R_{a_1} \oplus \cdots \oplus R_{a_n}$ of $a$ copies of $R$. Hence, there is a least integer $k$ such that $R^{(a)} = R_{k} \oplus \cdots \oplus R_k$ contains a copy $B$ of $C$. Since $B$ is indecomposable and injective, any two nonzero submodules of $B$ have nonzero intersection. Thus, if $k > 1$, then $B$ cannot have nonzero intersection with each component of $R_1$ of $R^{(a)}$. But if $B \cap R_k = 0$, for example, then the projection of $R^{(a)}$ on $R^{(a-1)} = R_{a} \oplus \cdots \oplus R_{a}$ maps $B$ monomorphically into $R^{(a-1)}$, which contradicts the definition of $k$. Thus, $k = 1$, so $B \subseteq R_1$, and $B$, being injective, is a direct summand of $R_1$. Thus, $\mathcal{C}$ is isomorphic to a direct summand of $R$.

3. Direct Sums of Finitely and Countably Generated Modules

A theorem of Kaplansky [6] states that if a module $M$ is a direct sum of countably generated modules then each direct summand of $M$ has the same property. We use this to prove the following theorem which generalizes the theorem of Cohen-Kaplansky [6] and Chase [5].

3.1. Theorem. If each module in $\mathfrak{M}_R$ is contained in a direct sum of finitely generated modules, then $R$ is right artinian.

Proof. If $M \in \mathfrak{M}_R$ is injective, then $M$ is a direct summand of each over-module, so $M$ is a direct sum of countably generated modules by Kaplansky’s theorem. Then Theorem 1.1 implies that $R$ is right noetherian. Now let $C$ be any cyclic module in $\mathfrak{M}_R$. Then $\mathcal{C}$ is contained in a direct sum of

Footnote: The hypothesis implies that each injective module is a direct sum of finitely generated modules. (Hint: combines the proof of 3.1 with Theorem 0.1.) In this case, $R$ is also left-artinian by Rosenberg-Zelinsky [20].
finely generated modules, so $C$ is finely generated by Proposition 2.4. Then $R$ is right artinian by Corollary 2.3.

3.2. **Corollary.** Let $R$ be a commutative ring. Then $R$ is artinian if and only if each injective $R$-module is a direct sum of finely generated modules.

**Proof.** One way follows from the theorem above. Conversely, let $R$ be a commutative artinian ring. Then, since $R$ is noetherian, an injective module $M$ is a direct sum of indecomposable modules. By a theorem of Macaia [17, p. 122, Lemma 2.1], an indecomposable injective module over $R$ is finely generated. This completes the proof.

If $R$ is a ring with the property of the theorem, then each finely generated module $C$ is contained in a direct sum of finely generated modules and then Proposition 2.4 implies that $C$ is finely generated. Expressed otherwise, if $C$ is a module of finite length, then $C$ has finite length. Rosenberg and Zelinsky [21] have shown that in general right artinian rings do not enjoy this latter property, consequently cannot have the former property.

Carol Walker [23] has generalized Kaplansky’s theorem as follows: If $M$ is a module which is a direct sum of modules each generated by $c$ elements, where $c$ is an infinite cardinal, then each direct summand of $M$ is a direct sum of modules each generated by $c$ elements. Using this theorem, we can generalize Theorem 1.1 as follows:

3.3. **Theorem.** A ring $R$ is right noetherian if and only if there exists a cardinal number $c$ such that each right $R$-module is contained in a direct sum of modules generated by $c$ elements.

**Proof.** If $R$ is right noetherian, Theorem 1.1 and the fact that every module is contained in an injective module gives the desired $c$. Conversely, suppose such a $c$ exists. Then the generalization of Kaplansky’s theorem above, together with Theorem 1.1 yields $R$ right noetherian.

4. **COGENERATORS**

In Section 5 we characterize a ring $R$ with the property that each injective module is projective. Since each module over $R$ is contained in an injective module, it then follows that each module is contained in a direct sum of copies of $R$. In particular, $R$ is then a cogenerator. This section is devoted to proving that if $R$ is a cogenerator and $R/\text{rad } R$ is semi-simple, then $R$ is self-injective. This last fact paves the way for the characterization mentioned above.
DEFINITION. A module $C$ in $\mathfrak{M}_R$ is a cogenerator in case it possesses one of the following equivalent properties:

(1) Each module $M \in \mathfrak{M}_R$ can be mapped monomorphically into a direct product of copies of $C$.

(2) If $M$ and $N$ are in $\mathfrak{M}_R$, and $f:M \to N$ a nonzero map, then there exists a map $g:N \to C$ such that the composition map $gf:M \to C$ is nonzero.

We also need the dual concept of generator.

A module $G$ in $\mathfrak{M}_R$ is a generator in case it possesses one of the following equivalent properties:

(1) Each module $M$ in $\mathfrak{M}_R$ is an epimorph of a direct sum of copies of $G$.

(2) If $M$ and $N$ are nonzero modules in $\mathfrak{M}_R$, and $f:M \to N$ a nonzero map, then there exists a map $g:N \to G$ such that the composition map $fg:G \to M$ is nonzero.

The equivalence of the statements (1) and (2) are well-known, and we omit the proofs. We also use, without proof, the following characterizations:

PROPOSITION. (1) An injective module $M \in \mathfrak{M}_R$ is a cogenerator in case $M$ contains a copy of each simple module in $\mathfrak{M}_R$. (2) A projective module $P \in \mathfrak{M}_R$ is a generator in case each simple module in $\mathfrak{M}_R$ is an epimorph of $P$.

We can now state and prove the theorem on cogenerators needed in the following section.

4.5. THEOREM. If $\mathfrak{M}_R$ has a finitely generated projective cogenerator $P$, and if $R$ is semi-simple, then $P$ and $R$ are injective in $\mathfrak{M}_R$.

Proof. The proof is facilitated by the use of the following two results on projective modules.

1. Let $P$ and $Q$ be finitely generated projective modules in $\mathfrak{M}_R$, and let $f = \text{rad } R$. Then $P/Pf \cong Qf/Q$ if and only if $P \cong Q$.  

2. Let $P$ be a finitely generated projective module in $\mathfrak{M}_R$, $A = \text{End}_R P$, $Q = \text{rad } A$, and $f = \text{rad } R$. Then the ring $\text{End}_Q P/Pf$ is isomorphic to the quotient ring $A/Q$.

I and II appear several places in the literature. For example, I is stated in [18, p. 218], and II can be deduced from [21, p. 57, Corollary 2.5].

If $U$ is any simple module in $\mathfrak{M}_R$, and if $f$ is its injection into its injective hull $\hat{U}$, then the cogenerator property of $P$ yields a map $g:U \to P$ such that $gf \neq 0$. It follows that $g$ is a monomorphism, since $\text{Ker}(g) \cap f(U) = 0$ implies $\text{Ker}(g) = 0$.

Since $R/f$ is semi-simple, $f = \text{rad } R$, there are only finitely many non-isomorphic simple modules $U_1, \ldots, U_n$. By what has just been proved, we
may assume $\tilde{U}_1, \ldots, \tilde{U}_n$ are contained in $P$. Since these modules are injective, they are direct summands of $P$, hence are finitely generated and projective along with $P$. Since $U_1, \ldots, U_n$ are non-isomorphic simple submodules of $P$, they are independent submodules, that is, the sum $U_1 + \cdots + U_n$ is direct. Consequently the sum $\tilde{U}_1 + \cdots + \tilde{U}_n$ is direct. Furthermore, $C = \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_n$, being a direct sum of projective injective modules, is projective and injective. Since $U_i$ is simple, $\tilde{U}_i$ is indecomposable (and injective), so $\Lambda_i = \text{End}_R \tilde{U}_i$ is a local ring. By II, $\Lambda_i/\tilde{Q}_i \cong \text{End}_R \tilde{U}_i / \tilde{U}_i J_i$, where $\tilde{Q}_i = \text{rad } \Lambda_i$, $i = 1, \ldots, n$. Since $\tilde{U}_i / \tilde{U}_i J_i$ is a simple module whose endomorphism ring is a division ring $\cong \Lambda_i / \tilde{Q}_i$, $\tilde{U}_i / \tilde{U}_i J_i$ must be a simple module. Since $\tilde{U}_i$ is finitely generated, I implies that $\tilde{U}_i / \tilde{U}_i J_i$ is isomorphic to $\tilde{U}_i / \tilde{U}_i J_i$ if and only if $\tilde{U}_i \cong \tilde{U}_i$. Since $U_i$ is the unique simple submodule of $\tilde{U}_i$, and since isomorphic modules have isomorphic injective hulls, we set that this occurs if and only if $U_i \cong \tilde{U}_i$, that is, if and only if $i = j$. Thus, $\tilde{U}_1 / \tilde{U}_1 J, \ldots, \tilde{U}_n / \tilde{U}_n J$ is a complete set of $n$ nonisomorphic simple modules. Expressed otherwise, each simple module is an epimorphism of $C = \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_n$. Since $C$ is projective, this implies that $C$ is a generator in $M_R$. Therefore, there exists an epimorphism $\sum C_i \rightarrow R$ of a direct sum of copies of $C$ onto $R$. Since $R$ is projective, this epimorphism splits, so $R$ is (isomorphic to) a direct summand of $\sum C_i$. Since $R$ is finitely generated, $R$ is a direct summand of finitely many copies of $C$. Then injectivity of $C$ implies that $\tilde{U}$ is a generator of $R$. Since $P$ is finitely generated and projective, the same argument shows that $P$ is also injective in $M_R$.

4.2. COROLLARY. If $R$ is a cogenerator in $M_R$, and if $R/\text{rad } R$ is semi-simple, then $R$ is right self-injective.

5. A CHARACTERIZATION OF QUASI-FROBENIUS RINGS

In this section we deduce the characterization of $QF$ (quasi-Frobenius) rings mentioned in the introduction.

If $X$ is a subset of a ring $R$, set $(X:0) = \{a \in R \mid xa = 0\}$, and $(0:X) = \{a \in R \mid ax = 0\}$. Any right (resp. left) ideal of $R$ of the form $(X:0)$ (resp. $(0:X)$) is a right (resp. left) annulet.

A ring $R$ is $QF$ in case: (1) each right ideal is a right annulet; (2) each left ideal is a left annulet; and (3) $R$ is right (or left) artinian.

Next [23], and Elsénig-Nakayama [9], show the equivalence of the following conditions:

1. $R$ is $QF$.
2. $R$ is left self-injective and right artinian.
3. For all $x \in R - \text{rad } R$, $\text{rad } (xR) = (xR)$.

4. $R$ is left self-injective and right finitely generated.
5. $R$ is left self-injective and right Noetherian.

6. $R$ is right self-injective and right finitely generated.
7. $R$ is right self-injective and right Noetherian.
5.1. Proposition. The following two conditions on a ring $R$ are equivalent:

(1) Each cyclic module in $\mathbb{M}_R$ is contained in a projective module in $\mathbb{M}_R$;

(2) Each right ideal of $R$ is the right annihilator of a finite subset of $R$.

When (1) or (2) holds, then each cyclic module is contained in a finitely generated free module.

Proof. (1) $\Rightarrow$ (2). Let $A$ be a right ideal of $R$, and imbed the cyclic module $R/A$ in a projective module, hence in a free module $F \in \mathbb{M}_R$. Since $R/A$ is cyclic, we may assume that $F$ has a finite free basis, that is, that $F \cong R^n$ for some $n$. If we write $1 + A = \langle x_1, \ldots, x_n \rangle \in R^n$, where $x_i \in R$, $i = 1, \ldots, n$, and if $r \in R$, then $r + A = \langle x_1 r, \ldots, x_n r \rangle$. Thus $r \in A$ if and only if $x_i r = 0$, $i = 1, \ldots, n$. Thus $A = \langle x_1, \ldots, x_n \rangle$.

(2) $\Rightarrow$ (1). Any cyclic module has the form $R/A$ for some right ideal $A$. Now $A = \langle X, 0 \rangle$, for some finite subset $X = \{x_1, \ldots, x_n\}$ of $R$. Then, the correspondence $r + A \rightarrow (x_1 r, \ldots, x_n r)$, defined for each coset $(r + A) \in R/A$, imbeds $R/A$ into the free module $R^n$. This also proves the last statement.

The following result is proved similarly.

5.2. Proposition. The following two conditions on a ring $R$ are equivalent:

(1) Each cyclic module in $\mathbb{M}_R$ is contained in a direct product of copies of $R$;

(2) Each right ideal of $R$ is a right annihilator.

It is fairly well known that a ring $R$ is QF if and only if the class of injective modules in $\mathbb{M}_R$ coincides with the class of projective modules in $\mathbb{M}_R$. Our characterization of QF rings is

5.3. Theorem. A ring $R$ is QF if and only if each injective right $R$-module is projective.

Proof. First assume that $R$ is QF, and let $M$ be any injective right $R$-module. Since $R$ is right artinian, it is right noetherian, so $M$ is a direct sum of indecomposable injective modules. Since a direct sum of projective modules is projective, we are reduced to the case where $M$ itself is indecomposable (and injective). Then $M$ is the injective hull $\hat{C}$ of any nonzero cyclic submodule $C$. By the first proposition above, $C$ is contained in a finitely generated free module $R^n$. Since $P_R$ is injective, so is $R^n$; therefore the imbedding of $C$ into $R^n$ can be extended to an imbedding of $M = \hat{C}$ into $R^n$. But $M$, being injective, is a direct summand of $R^n$, and therefore is projective.
Conversely, if each injective module in $\mathbb{M}_R$ is projective, then each module in $\mathbb{M}_R$ is contained in a free module, in particular, is contained in a direct sum of cyclic modules, so $R$ is right artinian by Theorem 3.1. But, as noted in the introduction to the preceding section, since each $M \in \mathbb{M}_R$ is contained in a direct product, in fact, direct sum, of copies of $R$, $R$ is a cogenerator in $\mathbb{M}_R$. Now $R/\rad R$ is semi-simple, so Cor. 4.2 implies that $R$ is right self-injective, so $R$ is $QF$ by the theorem of Ikeda.

The proof has the following consequence:

5.4. Corollary. An artinian ring $R$ is a cogenerator in $\mathbb{M}_R$ if and only if $R$ is $QF$.

5.5. Theorem. A ring $R$ is $QF$ if and only if each injective right $R$-module is a direct sum of cyclic modules which are isomorphic to principal indecomposable right ideals of $R$.

Proof. If $R$ is $QF$, and $M$ is injective, then $M$ is a direct sum of indecomposable injective modules $(M_i)_{i \in I}$. But each $M_i$ is projective by (5.3), and hence isomorphic to a principal indecomposable right ideal by (2.5). This proves one part. Since a direct sum of principal indecomposable right ideals is projective, the converse follows from (5.3).

We close out this section with a number of more or less obvious corollaries.

5.6. Corollary. A ring $R$ is $QF$ if and only if every right $R$-module is contained in a free right $R$-module.

5.7. Corollary. If every (injective) right $R$-module is contained in a direct sum of right ideals, then $R$ is $QF$.

For then every injective module is projective.

5.8. Corollary. If $R$ is left and right noetherian, and if $R$ is cogenerator in $\mathbb{M}_R$, then $R$ is $QF$.

Proof. If $R$ is right and left noetherian, then every finitely generated submodule of a direct product of copies of $R$ is contained in a free $R$-module. (Expressed otherwise, every finitely generated torsionless module is contained in a free module (see Bass [3, p. 477, (4.5)]). Hence, by (5.1), every right ideal of $R$ is an annulet, so $R$ is $QF$.

5.9. Corollary to 5.1. If $R$ is commutative then the following conditions are equivalent:

(1) Every cyclic module is contained in a projective $R$-module;
(2) Each ideal is the annihilator of a finite subset of $R$;
(3) $R$ is $QF$. 
Proof. (1) and (2) are equivalent for any ring $R$, by (5.1). If $R$ is $QF$, then every module can be embedded in a projective module, so (3) ⇒ (1).

Conversely, (2) implies every ideal is an annulet, so it remains to show that $R$ is artinian, or noetherian. Let $I$ be any ideal, and let $Q = (I : 0)$. Since $I$ is an annulet, $I = (Q : 0)$. By (2) there exist finitely many $x_1, ..., x_n \in R$ such that $\sum_{i=1}^{n} (x_i : 0) = Q$. Then $K = \sum_{i=1}^{n} x_i R$ is an ideal of $R$ and $(K : 0) = Q$. Since $K$ is an annulet, necessarily

$$I = (Q : 0) = K = \sum_{i=1}^{n} x_i R.$$  

Since every ideal $I$ of $R$ is therefore finitely generated, $R$ is noetherian, whence $QF$.

The same proof establishes

5.10. Corollary. The following conditions on a ring $R$ are equivalent:

1. Every cyclic right, and every cyclic left, $R$-module is contained in a projective $R$-module;
2. Every right ideal, and every left ideal, is the annihilator of a finite subset of $R$;
3. $R$ is $QF$.

5.11. Corollary. If $R$ is a ring, and if the injective hull of every cyclic right module, and the injective hull of every cyclic left module, is projective then $R$ is $QF$.

There exists a ring $R$ with precisely three right ideals $R \supset \text{rad } R \supset 0$ which is not left artinian. For this ring, $R$, $O$, and

$$\text{rad } R \cong R/\text{rad } R$$

are the only cyclic right $R$-modules. Thus, $R$ is an example of a ring which is not $QF$, yet every cyclic right $R$-module is contained in a free $R$-module, in fact, contained in $R$.

6. Completely Decomposable Modules

Let $Q = \sum_{i \in I} Q_i$ be a direct sum of indecomposable injective modules $(Q_i : i \in I)$. Such a module $Q$ will be called completely decomposable (c.d.). If $I$ is finite then $Q$ will be called finitely c.d. (A theorem of Azumaya [1, p. 119] states that each indecomposable injective submodule of the module $Q$ above is isomorphic to some $(Q_i :)$.)
In [26] Matlis asked: Is each direct summand $S$ of the module $Q$ (above), also c.d.? An affirmative answer is given by the Azumaya-Krell-Schmidt-Remark theorem when the index set $I$ is finite. Also, if $R$ is right hereditary, then $Q$ is injective by Theorem 0.1; $S$ is then injective, so $S$ is c.d. by Matlis' and Papp's Theorem 0.3. Below we give affirmative answers in some other special cases. We begin with a lemma.

6.1. Lemma. Let $Q$ be a direct sum of indecomposable injective modules \( \{Q_i \mid i \in I\} \), and let $S$ be a direct summand of $Q$, $Q = S \oplus T$. Then (1) If $M$ is a finitely generated submodule of $S$, then $S$ contains an injective hull $M$ of $M$, and $M$ is finitely c.d. (2) If $S_i$ is any direct summand of $S$, $S = S_i \oplus N$, and if $y \in S_i$, then $y \in S_i \oplus T_i$, where $T_i$ is a finitely c.d. summand of $N$.

Proof. (1) Let $x_1, \ldots, x_n$ generate $M$. Each $x_i$ is contained in the sum $P$ of finitely many of the $Q_i$'s, whence $M \subseteq P$. Since $P$ is injective, it contains an injective hull $P_i$ of $M$, into $S$ along $T$ fixes $M$, so is a monomorphism since $M$ is essential in $P_i$. The image of $P_i$ is clearly an injective hull $M$ of $M$. By the case cited above when the index set is finite, $P_i$, being a summand of $P$, is finitely c.d. Thus so is $M$, being isomorphic to $P_i$.

(2) Write $y = x + t$, with $x \in S_i$, $t \in N$. Applying (1) to the direct summand $N$ of $Q$, and the module $1t \subseteq N$, we obtain a direct summand $T_i$ of $N$ containing $1t$ which is finitely c.d., and $S_i \oplus T_i$ contains $y$ as required.

Note that (1) implies that the injective hull of any finitely generated submodule of a c.d. module is finitely c.d.

6.2. Proposition. Let $S$ be a direct sum of the c.d. module $Q$.
(1) If $S$ is countably generated, then $S$ is c.d.; (2) If $S$ is the injective hull of a finitely generated submodule, then $S$ is finitely c.d.; (3) If $S$ contains a finitely generated submodule $M$, such that $S/M$ is countably generated, then $S$ is c.d.

Proof. (1) Let $x_1, x_2, \ldots, x_n$ be a countable generating set of $S$. By the lemma, $S$ contains a chain $S_n \subseteq \cdots \subseteq S_1 \subseteq \cdots$ of direct summands such that $x_1 \ldots, x_n \in S_i$ for all $i$, and such that each $S_i$ is finitely c.d. But $S = \bigcup_{n=1}^{\infty} S_n$. Setting $S_0 = 0$, $S \cong \bigoplus_{n=1}^{\infty} S_n/S_n$ is clearly a direct sum of finitely c.d. modules, that is, $S$ is c.d.

(2) follows immediately from the lemma, and (3) is then a consequence of (1) and (2).

6.3. Corollary. Let $Q$ be a direct sum of (any number of) countably generated indecomposable injective modules. Then any direct sum $S$ of $Q$ is c.d.
Proof. By Kaplansky’s theorem, stated in Section 3, S itself is a direct sum of countably generated modules. Hence, it suffices to prove the corollary for the case S is countably generated. But this case follows from (1) of the last proposition.

6.4. Theorem. Let Q be an injective right R-module which is completely decomposable: 

\[ Q = \bigoplus_{i \in I} Q_i, \]

where \( \{Q_i : i \in I\} \) is a set of indecomposable (and injective) modules.

(1) If S is any submodule which is a direct sum \( \bigoplus_{j \in J} S_j \) of indecomposable injective submodules \( \{S_j : j \in J\} \), then S is injective, and \( |J| \leq |I| \).

(2) Any direct summand \( P \) of Q is completely decomposable.

Proof. For any subset A of I, let \( Q_A = \bigoplus_{a \in A} Q_a \), and let \( \pi_A \) be the projection of Q on \( Q_A \) having kernel \( Q_{-A} \). A homogeneous component of Q is defined to be a submodule \( Q_a \), where each \( A \) is the set of all those \( b \in I \) for which \( Q_b \) is isomorphic to some fixed \( Q_a \). Then \( Q = \bigoplus \bigoplus Q_a \), and if S is the submodule in the statement of (1), then also S is a direct sum of its homogeneous components: \( S = \bigoplus Q_a \).

(i) If \( h \in K \), then \( S_h \), being injective, is a direct summand of Q, and being indecomposable, is isomorphic to \( Q_a \) for some \( a \in I \). We first show:

(i) \( S_a \) is isomorphic to a submodule of \( Q_a \), the homogeneous component of Q determined by a. Suppose for the moment that \( S_a \cap \ker \pi_a \neq 0 \). If y is a nonzero element in this intersection, then the submodule T it generates is contained in \( S_{a'} \), where \( a' \) is a finite subset of K, and also \( T \subseteq Q_b \), where B is a finite subset of \( I - A \). (Recall that \( \ker \pi_a = Q_{-a} \)). Since \( S_{a'} \) (resp. \( Q_b \)) is injective, it contains an injective hull E (resp. \( F \)) of T. Since \( S_{a'} \) is finitely completely decomposable and is isomorphic, so is E. Since E and F are isomorphic, this means that E contains submodule G which is isomorphic to \( S_k \), hence to \( Q_k \), which is also isomorphic to \( Q_{a'} \) for some \( b \in I - A \). But, by the definition of \( A \), this is impossible, since \( A = \{ a \in I | Q_a \cong Q_k \} \). This contradiction shows that \( \ker \pi_a \cap S_a = 0 \), so \( \pi_a \) maps \( S_a \) monomorphically into \( Q_a \). Hereafter \( \pi_a \) denotes a fixed monomorphism \( S_a \to Q_a \).

We next show:

(ii) \( S_a \) is injective. If \( |A| \) is infinite, then \( Q_a \) is countably \( \Sigma \)-injective, hence by (0.2), \( Q_a \) is \( \Sigma \)-injective. This implies that \( S_a \) is injective. Next suppose \( |A| = n \) is finite. If \( |E| > n \), then \( S_a \) contains as a direct summand a submodule T which is a direct sum of \( n + 1 \) copies of \( Q_a \). Furthermore, T is then injective, so \( \pi_a(T) \) is a direct summand of \( Q_a \), which is a direct sum of \( n + 1 \) copies of \( Q_a \), violating the Krull-Schmidt-
Remark theorem. Therefore $|K| < \infty$, so $S_k$ is injective in this case too. This proves (ii), and also (ii) below.

(iii) If $W$ is a submodule of $Q$, and if $W$ is a direct sum of isomorphic indecomposable injective submodules, then $W$ is injective. (iii) is the homogeneous case of (1), that is, the case where $W = S = S_k$.

In order to simplify notation, let $X = Q_k$ and $U = \varphi_k(S_k) = \varphi_q$. Since $U \approx S_k$, $U$ is injective, so $X = U \oplus V$ for some submodule $V$. By Zorn's lemma, there exists a maximal independent set $P$ of indecomposable injective submodules of $V$. The sum $W$ of the submodules in $P$ is direct. Since $X = Q_k$ is homogeneous, so is $W$, and $W$ is therefore injective by (ii). Since $W$ is therefore a summand of $V$, in order to avoid contradicting maximality of $W$, necessarily $W = V$. Together the completely decomposable modules $U$ and $V$ yield a decomposition, of $X = U \oplus V$ into a direct sum of indecomposable modules, and the unique decomposition theorem [2] then implies that the number of indecomposable summands of $X = \varphi_k(S_k)$ is less than $|A|$, that is, $|K| \leq |A|$. Now, $S_k$ is isomorphic to a direct summand of $Q_k$, and therefore $S = \sum S_k$ is isomorphic to a direct summand of $Q$. This proves that $S$ is injective. Furthermore, since $J$ (resp. $P$) is the disjoint union of the various $K$ (resp. $A$), then $|K| \leq |A|$ implies that $|j| \leq |1|$. This proves (1), (2) is proved in the same way above we proved that $V$ is completely decomposable.

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