Direct Summands of Direct Products of Abelian Groups

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Baueslai and Blackburn have investigated direct summands of direct products (direct product = unrestricted direct sum) of Abelian groups in [1]. One question left open is whether or not the direct product of cyclic groups of orders $p, p^2, p^3, \ldots$ has a non-zero torsion-free direct summand. We will show that it does have such a summand, and indeed a maximum one. This will be an easy corollary of our main Theorem, which settles a more general question. Furthermore, generalizations of Baueslai and Blackburn's Theorem 2 and Theorem 4 will be immediate consequences of the methods we use.

Our methods are homological. More precisely, we will use some of Harrison's results in [3] which were proved homologically. For convenience, let us summarize the results of [3] that will be applied here. The word group will always mean Abelian group. The additive group of a ring has the word group of integer will be denoted $Q$ and $Z$ respectively. If $G$ is a group, $G_i$ will denote the torsion subgroup of $G$.

A reduced* group $G$ is cotorsion if $\text{Ext}(H, G) = 0$ for all torsion free groups $H$. A cotorsion group is adjusted if it has no non-zero torsion free direct summand. The following principal results we will note here and will use without further explicit reference to them.

(A) Every cotorsion group $G$ is uniquely the direct sum of a torsion free cotorsion group and an adjusted cotorsion group. Moreover, a cotorsion group $G$ is adjusted if and only if $G/\mathbb{G}$ is divisible.

(B) There is a one-to-one correspondence between all divisible torsion groups and all torsion free cotorsion groups. If $D$ is a divisible torsion group, the correspondence is $D \leftrightarrow \text{Hom}(\mathbb{Q}/\mathbb{Z}, D)$. If $G$ is torsion free cotorsion, the inverse of this correspondence is $G \leftrightarrow (\mathbb{Q}/\mathbb{Z}) \otimes G$.

(C) There is a one-to-one correspondence between all reduced torsion groups and all adjusted cotorsion groups. If $T$ is a reduced torsion group, the correspondence is $T \leftrightarrow \text{Ext}(\mathbb{Q}/\mathbb{Z}, T)$. If $G$ is an adjusted cotorsion group, then the inverse of this correspondence is $G \leftrightarrow G_i$. (Result (C) will not be used later and is noted here only as an analogy to result (B).)

(D) A torsion group is cotorsion if and only if it is of bounded order. (Harrison's remark on page 371 of [3] is incorrect. See [2], page 187.)

*) $G$ is reduced if it has no non-trivial divisible subgroup.
Now we are ready to state and prove our main theorem, which is an almost immediate consequence of Harrison's result in [3]. However, the consequences of this theorem are worthy of note.

**Theorem.** Let \( \{G_i\}_{i=1}^n \) be a set of groups, where each \( G_i \) is torsion. Then \( \prod_{i=1}^n G_i = G \) has a maximum torsion free direct summand \( H \), and \( H = 0 \) if and only if \( G/G_i \) is divisible. Furthermore, any non-zero torsion free direct summand of \( G \) is uncountable.

**Proof.** Let \( A \) be any torsion free group. Since each \( G_i \) is torsion, \( 0 = \text{Ext}(A, G_i) = \prod_{i=1}^n \text{Ext}(A, G_i) = \text{Ext}(A, \prod_{i=1}^n G_i) \), so that \( G \) is torsion. If \( G = H \oplus K \), then \( 0 = \text{Ext}(A, G) = \text{Ext}(A, H \oplus K) \cong \text{Ext}(A, H) \oplus \text{Ext}(A, K) \), and it follows that any direct summand of \( G \) is also a torsion. Let \( G = H \oplus K \), where \( H \) is torsion torsion free and \( K \) is adjusted torsion. Suppose \( H' \) is any torsion free direct summand of \( G \), and let \( G = H' \oplus K' \). Write \( K' = L \oplus M \), where \( L \) is torsion free and \( M \) is adjusted. Then \( G = (H' \oplus L) \oplus M \), and by uniqueness it follows that \( H = H' \oplus L \). Hence \( H \) is the maximum torsion free direct summand of \( G \). Suppose \( H \neq 0 \). Proving \( \phi \) onto \( H \) is a homomorphism of \( G \) onto \( H \) whose kernel contains \( G_i \). Hence \( H \) is a homomorphic image of \( G/G_i \). But \( H \) is nonzero reduced. Therefore \( G/G_i \) is not divisible. Conversely, suppose \( G/G_i \) is not divisible. Then \( G \) is not adjusted, and hence \( H = 0 \). Now let \( H' \) be any non-zero torsion free direct summand of \( G \). Since \( H' \) is torsion, \( H' \cong \text{Hom}(G/E, D) \), where \( D \) is some nonzero torsion divisible group. But \( \text{Hom}(Z(p^\infty), Z(p^\infty)) \) is the \( p \)-adic integers, which are uncountable. Hence \( H' \) is uncountable.

Now let us see what this theorem says in some special cases. First, let \( G_i \) be the cyclic group of order \( p^i \) for some fixed prime \( p \). Each \( G_i \) is certainly torsion. Let \( G = \prod_{i=1}^n G_i \) and let \( g \) be a generator of \( G_i \). It is easy to see that the element \( (g_1, g_2, g_3, \ldots) \) is not divisible by \( p \) modulo \( G_i \), so that \( G/G_i \) is not divisible. By our theorem, \( G \) has a maximum torsion free direct summand \( H \), \( H = 0 \), and any torsion free direct summand of \( G \) is uncountable.

More generally, let \( \{G_i\}_{i=1}^n \) be a set of groups such that each \( G_i \) is bounded order. Let \( n_1 \) be the exponent of \( G_1 \). We know that \( G \) will have a non-zero torsion free direct summand if and only if \( G/G_1 \) is not divisible, but what does this mean in terms of the \( n_i \)? The situation is this. The group \( G/G_i \) is not divisible if and only if there exists a prime \( p \) and exponents \( n_1, n_2, \ldots \) such that \( p \) divides \( n_i \). The proof that the existence of such a prime \( p \) implies \( G/G_i \) is not divisible is analogous to the one above that \( \prod_{i=1}^n G_i \) is not divisible modulo its torsion subgroup. Suppose that no such prime exists. Let \( (g_i) \) be an element of \( G_i \), and let \( p \) be a prime. We wish to find an element \( (l_i) \in G \) and an element \( (a_i) \in G \) such that \( (t_i) + p(b_i) = (g_i) \). Let \( a(x) \) denote the order of an element \( x \). Let \( a(x) = p^m a \), where \( m \) is a fixed integer such that \( m \geq 1 \). If \( m = 0 \), then \( a(x) = p^m a \), and it is well known that \( p^m a \) is divisible by \( p \) in \( G_1 \). In this case let \( a_i = 0 \) and let \( a_i \) be any element in \( G_i \) such that \( p b_i = a_i \). If \( m_1 = 1 \), let \( a_i \) and \( b_i \) be integers such that \( a_i = (a_i, p^m a) + (b_i, q) \). Now let \( a_i = (a_i, p^m a) + (b_i, q) \).
Our hypothesis implies immediately that \( \langle \xi \rangle \in G_1 \), and it is obvious that \( \langle \xi \rangle + p\langle \xi \rangle = \langle \xi \rangle \). Hence \( \langle \xi \rangle \) is divisible modulo \( G_1 \) by any prime \( p \), and thus by any integer. Therefore \( G/G_1 \) is divisible.

In particular, we see that if \( \{G_a\}_{a \in \mathcal{A}} \) is a set of cotorsion groups such that each \( G_a \) is of finite exponent \( n_a \), and such that \( (n_{a_1}, n_{a_2}) = 1 \) if \( a_1 \neq a_2 \), then \( G \) has no non-zero torsion free direct summands.

We have seen that a direct summand of a cotorsion group is cotorsion. Since a torsion group is cotorsion if and only if it is of bounded order, we get that a torsion direct summand of any direct product of cotorsion groups is of bounded order, and in particular, Theorem 2 of [1].

References


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