ÉTUDES SUR LES GROUPES ABÉLIENS
STUDIES ON ABELIAN GROUPS

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AN EXTENSION OF THE UM-RELATIVITY THEOREM

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CHAPTER I

INTRODUCTION.

1.1. TERMINOLOGY AND NOTATION.

All groups considered in this paper will be reduced p-primary Abelian groups for a fixed prime \( p \). If \( G \) is such a group and \( a \) is an ordinal number, \( p^a G \) is defined by induction as follows: \( p^0 G = G \), \( p^a G = p(p^{a-1} G) \) if \( a = \omega \), and \( p^a G = \bigcap_\alpha p^\alpha G \) if \( a \) is a limit ordinal.

The length of a reduced \( p \)-group \( G \), denoted by \( \ell(G) \), is the minimum ordinal \( a \) such that \( p^a G = 0 \). The symbols \( \Theta \) and \( \Theta' \) will denote direct sums of groups. With a few stated exceptions, the terminology will be as in [1] and [2].

Suppose \( G = \bigoplus H_j \). If \( J \subseteq I \) and \( J \neq \emptyset \), then the symbol \( \Theta(J) \) will mean \( \bigoplus H_j \). If \( H \) is a non-zero subgroup of \( G \), the cylinder of \( H \) (with respect to a given decomposition), denoted by \( H^\infty \), is defined by

\[ H^\infty = \{ (x_j) \mid x_j \in H_j \} \]

where \( H \) is the minimal subset of \( H^\infty \) such that \( H \subseteq H^\infty \).

If \( X \) is a set and \( \alpha \) is a cardinal number, then the cardinal numbers of \( X \) and \( \alpha \) will be denoted by \( |X| \) and \( |\alpha| \), respectively. The first infinite ordinal will be denoted by \( \omega \), and the first non-cardinal ordinal by \( \omega' \), with \( N_\omega = |\omega| \) and \( N_{\omega'} = \omega \).

If \( a \) is an ordinal, a subgroup \( H \) of \( G \) is \( p^a \)-pure (sometimes called \( p^a \)-weakly-pure) in \( G \) if \( p^a G \cap H = p^a H \) for all ordinals \( a \leq \omega \).

1.2. UMA'S THEOREM.

If \( G \) is a reduced Abelian \( p \)-group, one can define a function \( f_p \) from ordinals to ordinals by \( f_p(a) = \dim \left( (p^a G)^2 / (p^{2a} G) \right) \) for any ordinal number \( a \), where \( \dim \) means dimension as a vector space over the \( p \)-element field. This function is called the UMA function of \( G \), and its values are rather loosely called the UMA invariants of \( G \), with \( f_p(a) \) being referred to as the \( a \)-th UMA invariant of \( G \) invariant because isomorphic groups are easily seen to have identical UMA functions. UMA's famous result is that the converse of this last statement holds for a certain class of groups.

\[ ^1 \text{Portions of this research were supported by NFS-GP-6596.} \]
THOMAS [11]. Two countable reduced Abelian $p$-groups are isomorphic if and
only if they have the same Ulm invariants.

It is not surprising that this is referred to as "the most celebrated result
in the theory of infinite abelian groups" [1] since it assures that the whole
structure of one of these groups is determined by a family of cardinal numbers.

Complementary to Ulm’s theorem is the existence theorem proved by Zipkin [15]
in 1955, which describes the Ulm function of a countable reduced $p$-group, and says
roughly that any function which could possibly be the Ulm function of such a
$\mathbb{Z}$-group actually is. Zipkin also proved a stronger form of Ulm’s theorem, stating
not only that two countable reduced $p$-groups $A$ and $B$ with the same Ulm inver-
siants are isomorphic, but also that for any ordinal $\alpha$, any isomorphism from
$p\mathbb{Z}^\alpha$ onto $p\mathbb{Z}^\beta$ can be extended to an isomorphism from $A$ onto $B$.

I.D. KOELLENS THEOREM

In 1950 Koellett proved the following extension of Ulm’s theorem:

THEOREM [16]. If the reduced primary Abelian groups $G$ and $H$ are direct
sums of countable groups, then $G$ and $H$ are isomorphic if and only if their
Ulm functions are equal.

Recently much shorter proofs [17] and [20] of this theorem have appeared
than the one given by Koellett. In fact, the proof is accomplished in [20] "without
group theory". However, Koellett’s paper [16] contains another major result. He pro-
ved an existence theorem for direct sums of countable $p$-groups analogous to Zipkin’s
existence theorem for countable $p$-groups. This theorem will be explicitly stated in
Chapter IV.

Hill and Neifer [20] have recently extended Zipkin’s stronger form of Ulm’s
Theorem, which was mentioned at the end of Section I.A, to the class of direct sums
of countable $p$-groups.

II. RELATED RESULTS

Other extensions of Ulm’s Theorem, and results closely related to Ulm’s
Theorem, have also appeared. We first describe the ones presented by Mitchell in
1958. A $1$-group is a primary Abelian group in which every subgroup maximal with
respect to disjointness from $p\mathbb{Z}$ is a direct sum of cyclic groups.

THEOREM [18]. Let $G$ and $H$ be two Abelian $p$-groups such that $G$ and $H$
are $1$-groups and $p^mG$ has a countable basis subgroup. Then $G$ and $H$ are
isomorphic if and only if $p^mG \cong p^mH$ and $G$ and $H$ have the same Ulm invariants.

Another related result was recently presented by Hill and Neifer.
An extension of the Ulm-Kolettis theorem

THEOREM [4]. Let \( \kappa \) be a countable limit ordinal and suppose that \( \xi \) is a
ordinary group such that \( 0/\xi^0 \) is a direct sum of countable groups. Then a primary
\( \xi \) is isomorphic to \( \xi \) if and only if \( \xi/\xi^0 \leq \xi/\xi^0 \) and \( \xi/\xi^0 \cong \xi^0 \).
Indeed, any isomorphism between \( \xi/\xi^0 \) and \( \xi/\xi^0 \) can be extended to an isomorphism
between \( \xi \) and \( \xi \).

It will be noted that neither of the above results is an extension of
Kolettis' Theorem, which is our objective. Such an extension involves finding
a new class of groups which properly contains the class of direct sums of countable
groups such that two groups in the new class are isomorphic if and only if
they have the same Ulm invariants. However, it is rather obvious that even this
is not enough for a reasonable extension of Kolettis' Theorem. Consider the class of
direct sums of countable groups, and form a new class by throwing in a group
of length \( \omega + 1 \). This new class will clearly "satisfy" Ulm's Theorem, but it is
not a meaningful extension of Kolettis' Theorem. We will discuss in the next
section a possible "ideal" extension of Ulm's Theorem.

1.5 TOTALLY PROJECTIVE GROUPS.

Nakane [5] has recently defined a new class of Abelian groups, called the
totally projective groups. If \( \alpha \) is an ordinal, a group \( A \) is \( \alpha \)-projective if
\( \alpha \)-Ext(\( A,C \)) = 0 \( \alpha \) for all groups \( C \). An Abelian \( \pi \)-group \( A \) is \( \pi \)-projective if
it is reduced and \( \pi \)-projective for every ordinal \( \alpha \).

Nakane's results concerning the class of totally projective groups (henceforth
denoted by \( P \)) are extensive. He shows that \( P \) is closed under the operations of
forming arbitrary direct sums and taking direct summands, and that, for each
ordinal \( \alpha \) \( \alpha \)-\( P \), if and only if \( \pi \alpha \)-\( P \) and \( \alpha \)-\( P \). He also shows that
the class of direct sums of countable reduced \( \pi \)-groups is precisely the class of
totally projective groups of length less than or equal to \( \alpha \). It was this last
result, together with a theorem of Hill and Negiben [1, Theorem 4] which sugges-
ted an attempt to extend Ulm's Theorem to the class \( P \) of totally projective
groups.

This would be the "ideal" extension of Ulm's Theorem mentioned in the previous
section. We pointed out there that it is easy to get worthless extensions of
Ulm's Theorem. It seems clear that a reasonable extension should involve a "nice"
class of groups, in the sense of being closed under the formation of sums and
summands, etc. If it were true, as we conjecture, that two totally projective
groups with the same Ulm invariants are isomorphic, then the extension of Ulm's
Theorem to the class \( P \) of totally projective groups would be the best possible
extension in the sense that \( P \) would be the only class of groups satisfying

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In this chapter we present a result which is a partial solution of the problem of extending Ulm's Theorem to the class of totally projective groups. The class of groups involved in our extension is the class of totally projective groups of length less than \( \omega_1 \). This class is closed under the formation of finite direct sums and taking direct summands, and if \( A \) is in the class and \( a \) is an ordinal, then \( p^aA \) and \( \langle p^a \rangle A \) are in the class. Conversely, if \( p^aA \) and \( \langle p^a \rangle A \) are in the class, then \( A \) is in the class if and only if \( \lambda(A) < \omega_1 \). As we mentioned, the direct sum of countable groups is precisely the totally projective groups of length less than or equal to \( \omega_1 \), so our result is actually a considerable extension of Kulshtein's Theorem. In fact, it is an extension of the stronger form of Kulshtein's Theorem mentioned in Section 1.3.

**Theorem 1.** If \( A \) and \( B \) are totally projective groups of length less than \( \omega_1 \) with the same Ulm invariants, then \( A \) and \( B \) are isomorphic. Moreover, for any limit ordinal \( \alpha \), any isomorphism from \( p^\alpha A \) onto \( p^\alpha B \) can be extended to an isomorphism from \( A \) onto \( B \).

The proof will be presented in several steps. We will induct on \( \lambda(A) = \lambda(B) \), first showing that \( p^aA \cong p^aB \) and \( p^\beta A \cong p^\beta B \). To prove that \( A \cong B \), we will then reduce to the case where \( |A| = |B| \). After making another induction, we will prove the theorem for the case where \( p^aA \) and \( p^\beta B \) are direct sums of cyclic groups. The proof of the second part of the theorem involves an application of Lenz's Lemma.

We now proceed with the proof of the first part of Theorem 1, that \( A \) and \( B \) are isomorphic. We induct on \( \lambda(A) = \lambda(B) \). If \( \lambda(A) \leq \omega_1 \), then \( A \) and \( B \) are direct sums of countable groups [3, Theorem 2.10], so we are finished by Kulshtein's Theorem. Thus we may assume \( \lambda(A) > \omega_1 \). Then \( p^aA \) and \( p^\beta B \) are totally projective [2, Props. 4,5], \( p^aA \) and \( p^\beta B \) have the same Ulm invariants, and
\( \lambda(p^2A) = \lambda(p^2B) \times \lambda(A) = \lambda(B) \) since \( \lambda(A) = \lambda(B) \times q_A \). Thus we apply the induction hypothesis to conclude that \( p^2A \preceq p^2B \). \( A/p^2A \) and \( B/p^2B \) are totally projective \([B, \text{Prop. 2.6}]\), so are direct sums of countable groups \([B, \text{Theorem 2.12}]\). Since \( A/p^2A \) and \( B/p^2B \) have the same Ulm invariants, \( A/p^2A \cong B/p^2B \) by Koelink's Theorem.

Now \( A \) is a direct summand of a direct sum of groups of cardinality at most \( 2^{\aleph_0} \) \([B, \text{Theorem 2.12}]\), and likewise for \( B \). Thus \( A \) and \( B \) are direct sums of groups of cardinality at most \( 2^{\aleph_0} \) \([C, \text{Theorem 4.3}]\).

Let \( A = \bigoplus_{i \in I} A_i \) and \( B = \bigoplus_{j \in J} B_j \), where \( |A_i| \leq \aleph_1 \) and \( |B_j| \leq \aleph_1 \) for each \( i \in I \). Then \( \forall i \) there is a partition \( (I_i : u \in H) \) of \( I \) into subsets \( I_i \) of cardinality at most \( \aleph_1 \) such that \( A[I_i] \) has the same Ulm invariants as \( B[I_i] \).

For each \( u \in H \). But \( |A[I_u]| \leq \aleph_1 \) and \( |B[I_u]| \leq \aleph_1 \) for each \( u \in H \), and \( A[I_u] \) and \( B[I_u] \) are totally projective \([B, \text{Prop. 2.6}]\), so we may assume that \( |\Delta| = |\Delta| = |\Delta| \), since \( A = \bigoplus_{i \in I} A[I_i] \) and \( B = \bigoplus_{j \in J} B[J_j] \).

We now have \( A/p^2A \cong B/p^2B \), \( p^2A \cong p^2B \), and \( |A| = |\Delta| = |\Delta| \), and we wish to show \( A \cong B \). We will actually prove any isomorphism from \( p^2A \) onto \( p^2B \) can be extended to an isomorphism from \( A \) onto \( B \). First we will reduce to the case where \( p^2A \) and \( p^2B \) are direct sums of cyclic groups.

Let \( f \) be an arbitrary isomorphism from \( p^2A \) onto \( p^2B \), and let \( k \) be a basic subgroup of \( p^2A \). Then \( L = f(k) \) is a basic subgroup of \( p^2B \). Thus there exist subgroups \( G \) of \( A \) and \( H \) of \( B \) such that \( A/k \cong G/K \), \( B/k \cong H/L \), \( G \cong K \cong f(k) \), and \( H \cong L \).

Clearly \( G \times k \) is maximal with respect to \( G \cap p^2A = k \) and \( k \in \text{Ker}(f) \). Thus \( p^2A \cap k \cong p^2B \cap k \). Similarly \( p^2A \cap k \cong p^2B \cap k \). Thus \( A/k \cong G/K \) isomorphic onto \( B/k \cong H/L \). Hence \( f(k) \) could be extended to an isomorphism \( f \) from \( G \) onto \( H \). We could define \( f : A \rightarrow B \) by \( f(ae) = f(a) + f(e) \), where \( a \in p^2A \) and \( e \in K \). If \( a = a' + a'' \), then \( \lambda_a = \lambda_{a'} + \lambda_{a''} \). Thus \( f \) is well-defined. It is easy to see that \( f \) is an isomorphism from \( G \) onto \( H \) and that \( f \) extends \( f \).

Thus we may assume that \( p^2A \) and \( p^2B \) are direct sums of cyclic groups. We have \( A/p^2A \cong B/p^2B \), \( p^2A \cong p^2B \), and \( |A| = |\Delta| = |\Delta| \), and we wish to show
that any isomorphism from \( p^0 A \) onto \( p^0 B \) can be extended to an isomorphism from \( A \) onto \( B \).

Let \( f \) be an arbitrary isomorphism from \( p^0 A \) onto \( p^0 B \). We will extend \( f \) to an isomorphism \( F \) from \( A \) onto \( B \).

Let \( C = p^0 A \times_{\mathbb{Z}_2} p^0 \mathbb{Z}_2 \) (where \( p^0 \mathbb{Z}_2 \) is the cyclic group generated by \( x \)). For each \( k \in \mathbb{N} \) (the set of natural numbers) let \( J_k = \{ j \in \mathbb{N} : D(x_j) \subset p^k \} \), and let \( X_k = \bigoplus_{j \in J_k} X_j \). Then \( C = X_k \).

Write \( A/C = \bigoplus_{\alpha \in A/C} B/\alpha \) and \( B/\alpha \) as \( \bigoplus_{\gamma \in B/\alpha} B/\gamma \), where for each \( \alpha \in A/C \) and \( \beta \in B/\alpha \), \( \beta/\alpha \) is countable and \( A_{\alpha}/C \cong B_{\beta}/\alpha \).

Since \( \{ \alpha \} = \{ \alpha \} \), we can index the elements of \( I \) by the ordinals less than \( \omega \), so that \( I = \{ \alpha \} \).

We will construct by transfinite induction a collection \( \{ I_j : j < \alpha \} \) of

pairwise disjoint countable subsets of \( I \), indexed by the ordinals less than \( \omega \).

We proceed to construct \( I_0 \), for each \( x \in \mathbb{N} \), \( p^0 A_0 \cap X_0 \) is countable.

To see this, let \( x_1, x_2 \in p^0 A_0 \setminus X_0 \). Then \( x_1 = p^0 y_1 \) and \( x_2 = p^0 y_2 \) where \( y_1, y_2 \in A_0 \setminus X_0 \). If \( x_1 \in X_0 \), then \( A_0 = p^0 X_0 \) and \( x_1 \in X_0 \), so \( A_0 \cap X_0 \) is countable.

Thus different elements of \( p^0 A_0 \cap X_0 \) determine different elements of \( A_0/C \), so \( p^0 A_0 \cap X_0 \) is countable since \( A_0/C \) is countable.

Define \( C(0) = \bigoplus_{\alpha \in A_{0}/C} B_{\alpha}/\alpha \), where the cylinders are with respect to the decomposition \( C = \bigoplus_{\alpha \in A_{0}/C} B_{\alpha}/\alpha \). It is evident that \( C(0) \) is countable. Let \( \{ \alpha \} \) be a set consisting of one element from each non-zero coset of \( A_{0}/C \). Let \( \alpha \) be a countable ordinal such that \( \alpha \leq \tau(1) \) and \( p^0 \alpha \subset A_{0}/C \) for each \( n \in \mathbb{N} \) (where \( p^0 \alpha \) denotes the image of \( \alpha \)).

Then \( C_{\alpha} \), Theorem 7 there exists a countable \( p^\alpha(1) \)-pure subgroup \( T_1 \) of \( B \) such that \( \tau(C(\alpha)) \subset T_1 \).

Since \( T_1 \) is countable, there exists a countable subset \( \{1\} \) of \( I \) such that \( I_1 \in \{1\} \) and \( T_1 \in \tau(1) \). Since \( B = \bigoplus_{\alpha \in A_{0}/C} B_{\alpha}/\alpha \), we are using \( s(C(1)) \) to denote \( \bigoplus_{\alpha \in A_{0}/C} \tau(C(\alpha)) \).

Define \( \mathbb{H} = p^0 \tau(0) \cap \tau(1) \), where the cylinders are with respect to the decomposition \( \tau(C(1)) = \bigoplus_{\alpha \in A_{0}/C} \tau(C(\alpha)) \). Then \( \mathbb{H} \) is countable since \( B \tau(1) \mathbb{H} = \bigoplus_{\alpha \in A_{0}/C} \tau(C(\alpha)) \) is countable. If we define \( C(1) = p^\alpha(D(1)) \), then
since \( f(C(0)) \in p^{(1)}[\mathcal{T}(1)] \subset p^{N}[\mathcal{T}(1)] \), we have \( f(C(0)) \subset C(1) \) so \( C(0) \subset C(1) \). Let \( \{a^{(2)}_n : n \in \mathbb{N}\} \) be a set consisting of one representative from each non-zero coset of \( A[\mathcal{T}(1)]/f(C) \), and let \( \tau(2) \) be a countable ordinal such that \( \tau(1) \times \tau(2) \) and \( \beta_2(b^{(2)}_n) \leq \tau(2) \) for each \( n \in \mathbb{N} \).

The group \( C(1) \cup \bigcup_{n \in \mathbb{N}} a_n^{(2)} \) is a countable subgroup of \( A \), so \( \{0, \text{Theorem 2}\} \) there exists a countable \( p^{(2)} \text{-pure subgroup} \) \( F_0 \) of \( A \) such that \( C(1) \cup \bigcup_{n \in \mathbb{N}} a_n^{(2)} \subset F_0 \).

Since \( F_0 \) is countable, there exists a countable subgroup \( X(2) \) of \( I \) such that \( X(2) \subset f(2) \) and \( F_0 \subset X(2) \). Then \( C(2) \subset p^{(2)}[X(2)] \cap F_0 = p^{(2)}[X(2)] \subset p^{(2)}[I(2)] \).

Define \( C(2) = \bigoplus_{n \in \mathbb{N}} [p^{(2)}[X(2)] \cap X_n] \), where the cylinders are with respect to the decomposition \( C = \bigoplus_{n \in \mathbb{N}} X_n \). Then \( C(2) \) is countable since \( A[X(2)]/C = \bigoplus_{n \in \mathbb{N}} A[X_n] \) is countable, and \( C(1) \subset C(2) \) since \( C(1) \subset p^{(2)}[X(2)] \cap p^{(2)}[I(2)] \). Let \( \{a_n^{(2)} : n \in \mathbb{N}\} \) be a set consisting of one representative from each non-zero coset of \( A[X(2)]/C \), and let \( \tau(3) \) be a countable ordinal such that \( \tau(2) \times \tau(3) \) and \( \beta_3(a_n^{(2)}) \leq \tau(3) \) for each \( n \in \mathbb{N} \).

Then \( \{0, \text{Theorem 2}\} \) there exists a countable \( p^{(3)} \text{-pure subgroup} \) \( F_0 \) of \( B \) containing \( f(C(2)) \cup \bigcup_{n \in \mathbb{N}} a_n^{(2)} \), let \( \tau(3) \) be a countable subset of \( I \) such that \( X(2) \subset X(3) \) and \( F_0 \subset X(3) \). Then \( f(C(2)) \subset p^{(3)}[X(3)] \cap F_0 = p^{(3)}[X(3)] \subset p^{(3)}[I(3)] \).

Define \( C(3) = \bigoplus_{n \in \mathbb{N}} [p^{(3)}[X(3)] \cap X_n] \), where the cylinders are with respect to the decomposition \( C = \bigoplus_{n \in \mathbb{N}} X_n \). Let \( C(3) = \bigoplus_{n \in \mathbb{N}} A[X_n] \). Then \( C(3) \) is countable and \( C(2) \subset C(3) \).

Continuing in this manner by induction, we get a chain \( \{I(n) : n \in \mathbb{N}\} \) of countable subsets of \( I \).

Define \( C_0 = \bigcup_{n \in \mathbb{N}} C(n) \). Let \( \tau(0) \) be the smallest ordinal such that \( \tau(0) > \tau(n) \) for each \( n \in \mathbb{N} \). Define \( C_0 = \bigoplus_{n \in \mathbb{N}} [p^{(0)}[I(0)] \cap X_n] \). Then \( C_0 \) is countable since \( A[I_0]/C = \bigoplus_{n \in \mathbb{N}} A[X_n] \) is countable, let \( C = C_0 \otimes \Xi_0 \), where \( \Xi_0 = \Xi_0 \). We note the following facts:

(1) \( C_0 = \bigcup_{n \in \mathbb{N}} C(n) \).

Let \( j \in J \) such that \( j \in C(n) \). Since \( C(0) \subset C(1) \subset \ldots, x_j \in C(n) \) for
some even $n$, so $x_j \in \mu^n(I[n]) \cap \gamma_i(x_i)$ for some $k \in \mathbb{N}$. Thus there is some $e \in \mathbb{N}$ such that $\mu_1 \oplus 0$ and $\mu_1 \oplus \mu^n(I[n]) \cap \gamma_i(x_i)$, so $x_i \in \mu^n(I[n]) \cap \gamma_i(x_i) \oplus 0$. 

Let $j \in \mathbb{N}$ such that $x_j \in \gamma_i(x_i)$, then $x_j \in \mu^n(I[n]) \cap \gamma_i(x_i)$ for some $k \in \mathbb{N}$ so there is some $e \in \mathbb{N}$ such that $\mu_1 \oplus 0$ and $\mu_1 \oplus \mu^n(I[n]) \cap \gamma_i(x_i)$. Then $\mu_1 \oplus \mu^n(I[n]) \cap \gamma_i(x_i)$ for some even $n \in \mathbb{N}$, so $x_i \in \mu^n(I[n]) \cap \gamma_i(x_i) \oplus 0$.

(2) $\gamma_i(x_i)$ is pure in $A[I[n]]$.

Let $\gamma \in \gamma_i(x_i)$, $\gamma \neq 0$. Then $\gamma = y_{a_1}y_{a_2}\ldots y_{a_k}$, where $y_{a_1} \neq 0$ for $t = 1, 2, \ldots, k$. Thus $y_{a_1}(\gamma) = y_{a_1}(y_{a_2}\ldots y_{a_k}) = y_{a_1} = y_{a_2}\ldots y_{a_k}$.

For $t = 2, 3, \ldots, k$, since $y_{a_1} \neq 0$ and $y_{a_1}(\gamma) = 0$, we have $y_{a_1} = y_{a_1}(y_{a_2}\ldots y_{a_k}) \neq 0$.

$C_0$, so $y_{a_2}\ldots y_{a_k}$, $y_{a_1} = y_{a_2}\ldots y_{a_k}$, $y_{a_1} \neq 0$.

Thus $\gamma_i(x_i)$ has the same height in $F_0$ as in $A[I[n]]$, so $F_0$ is pure in $A[I[n]]$.

(3) $p^n\mu^n(I[n]) \cap \gamma_i(x_i)$. 

Let $x \in \gamma_i(x_i)$, then for some $x \in \mathbb{N}$, $\mu_1 \oplus \gamma_i(x_i)$ for all $x \in \mathbb{N}$. Let $x \oplus 0$ be odd such that $x \oplus 0 \in \mathbb{N}$. Then $x \in C(0) \cap \mu^n(I[n]) \cap \gamma_i(x_i) \cap \mu^n(I[n]) \cap \gamma_i(x_i)$.

So $x \in \mu^n(I[n]) \cap \gamma_i(x_i)$. 

Suppose $x \in A[I[n]] \cap \gamma_i(x_i)$, then show $x \notin \gamma_i(x_i)$. If $x \in \gamma_i(x_i)$, then $x \in \gamma_i(x_i)$ with $c \in \gamma_i(x_i)$ and $d \in \gamma_i(x_i)$ since $x \in \gamma_i(x_i)$, $c$ has finite height in $A[I[n]]$ since $p^n\mu^n(I[n])$ is pure in $A[I[n]]$. 

Thus $t_0 \geq 0$, since $c \in \gamma_i(x_i)$ with $d \in \gamma_i(x_i)$, $c$ has finite height in $A[I[n]]$, so $x$ has finite height in $A[I[n]]$, so $x \notin \gamma_i(x_i)$.

If $x \notin \gamma_i(x_i)$, then $x \in \gamma_i(x_i)$ for some even $n \in \mathbb{N}$, since $t_0 \geq 0$, $x \in \gamma_i(x_i)$. Thus $x \in \gamma_i(x_i)$ for some $e \in \mathbb{N}$ and $c \in \mathbb{N}$, where $x_{(a)}$ is a member of the set of representatives of non-zero coasts of $A[I[n]]$.

$b_1(x_{(a)}) \leq t_0 + 1$, so $b_1(x_{(a)}) = b_1(x_{(a)}) + t_0 + 1$, so $x \in \mu^n(I[n])$. 

(4) $p^n\mu^n(I[n]) \cap \gamma_i(x_i)$. 

Now $p^n\mu^n(I[n]) \cap \gamma_i(x_i)$, so $x \in \gamma_i(x_i)$ with $c \in \gamma_i(x_i)$. Thus $x \in \gamma_i(x_i)$ with $c \in \gamma_i(x_i)$, so $x \in \gamma_i(x_i)$. Then $x \in \gamma_i(x_i)$ for some even $n \in \mathbb{N}$, so $x \in \gamma_i(x_i)$ for some $e \in \mathbb{N}$ and $c \in \mathbb{N}$, so $x_{(a)} = x_{(a)} \in \mu^n(I[n]) \cap \gamma_i(x_i) \cap \mu^n(I[n]) \cap \gamma_i(x_i) \cap \gamma_i(x_i)$. Thus
\[ x^C = (x^n)^C = \left(\frac{y^A(x^n)^C}{C'}\right)^C \quad \text{so} \quad y^A(x^n)^C = \left(\frac{y^A(x)^C}{C'}\right)^C. \]  

The other inclusion is obvious.

(5) \( A[x^n]/(y^A(x^n)^C)^C \) is a direct sum of cyclic groups.

\[ A[x^n]/(y^A(x^n)^C)^C = \text{countable}, \quad \text{so} \quad A[x^n]/(y^A(x^n)^C)^C = \sum (A[x^n]/(y^A(x^n)^C)^C) = (A[x^n]/(y^A(x^n)^C)^C) = A[x^n]/(y^A(x^n)^C)^C. \]

(6) \( E_0 \) is a direct summand of \( A[x^n] \).

(\#) \( y^A(x^n)^C = A[x^n]/(y^A(x^n)^C)^C \) is pure in \( A[x^n]/(y^A(x^n)^C)^C \), and the quotient is a direct sum of cyclic groups by (5), so \( \left(\frac{y^A(x^n)^C}{(y^A(x^n)^C)^C}\right)^C \) is a direct sum of cyclic groups. Thus, \( E_0 \) is a direct summand of \( A[x^n] \).

Let \( A[x^n] = E_0 \oplus B[x^n] \) and \( y^A(x^n)^C = \text{countable} \). Then \( \sigma_0 \mathcal{E}_0^C = \sigma_0 \mathcal{E}_0^C \mathcal{E}_0^C = \sigma_0 \mathcal{E}_0^C \mathcal{E}_0^C = \sigma_0 \mathcal{E}_0^C \mathcal{E}_0^C \mathcal{E}_0^C = \cdots \). Since \( (\mathcal{E}_0^C)^C \mathcal{E}_0^C = \mathcal{E}_0^C \mathcal{E}_0^C = \mathcal{E}_0^C \mathcal{E}_0^C \mathcal{E}_0^C = \cdots \), there is a unique \( s \in \mathcal{E}_0 \) such that \( \mathcal{E}_0^C \mathcal{E}_0^C = \sigma_0 \mathcal{E}_0^C \mathcal{E}_0^C \mathcal{E}_0^C = \cdots \), and \( A[x^n] = (\sigma_0 \mathcal{E}_0^C)^C \mathcal{E}_0^C = (\sigma_0 \mathcal{E}_0^C)^C \mathcal{E}_0^C = (\sigma_0 \mathcal{E}_0^C)^C \mathcal{E}_0^C = \cdots \).

Let \( \mathcal{E}_0^C \) be an isomorphism from \( \mathcal{E}_0^C \mathcal{E}_0^C = \sigma_0 \mathcal{E}_0^C \mathcal{E}_0^C \mathcal{E}_0^C = \cdots \) onto \( \mathcal{E}_0^C \mathcal{E}_0^C \mathcal{E}_0^C = \sigma_0 \mathcal{E}_0^C \mathcal{E}_0^C \mathcal{E}_0^C = \cdots \).

We will show that \( y^A(x^n)^C = \mathcal{E}_0 \).

Since \( A[x^n] = A[x^n] \mathcal{E}_0 \), \( y^A(x^n)^C \subset y\mathcal{E}_0 = \mathcal{E}_0 \). We will show by induction that \( C \subset y^A(x^n)^C \) for each ordinal \( u \in \mathbb{C} \).

\[ x^C = (x^n)^C = \left(\frac{y^A(x^n)^C}{C'}\right)^C \quad \text{so} \quad y^A(x^n)^C = \left(\frac{y^A(x)^C}{C'}\right)^C. \]
It is clear from the construction that if \((t_a : a \in C)\) is a partition of \(I\), thus \(A/C = \bigoplus_{a \in C} [1]_{a}/C\) and \(B/C = \bigoplus_{a \in C} [1]_{a}/C\), for each \(a \in C\) we have an isomorphism \(r_a : A[a] \to B[a]\) such that \((r_a)_{a} : I \to I\). Since \(A = \bigoplus_{a \in C} A[a]\) and \(B = \bigoplus_{a \in C} B[a]\), and since \((r_a)_{a} : I \to I\) for each \(a \in C\), the function \(\bar{r} : I \to I\) defined by \(\bar{r}(t_a) : I \to I\) is a well-defined isomorphism from \(A\) onto \(I\), and \(\bar{r}\) extends \(r\).

We have now proved the first part of Theorem 1: that two totally projective groups \(A\) and \(B\) of length less than or with the same Ulm invariants are isomorphic. We now assume that \(a\) is a limit ordinal and that \(f\) is an arbitrary isomorphism from \(A[a]\) onto \(B[a]\), and we wish to extend \(f\) to an isomorphism from \(A\) onto \(B\). We do this by induction on \(a\). If \(a = 0\), then \(A[a]\) is totally projective \([20, \text{Prop. 2}, \text{C.1}]\) and of length less than or \(\geq\), so \(A[a]\) is a direct sum of countable groups \([1, \text{Theorem 2.1}]\). Thus \(f\) can be extended to an isomorphism from \(A\) onto \(B\) \([1, \text{Theorem 2.1}]\).

Suppose \(a = \omega\). Then we must show that any isomorphism from \(A[a]\) onto \(B[a]\) can be extended to an isomorphism from \(A\) onto \(B\). We have shown in the course of the above proof that this is so provided that \([a] \times \omega = [b] \times \omega\), we have 

\[A = \bigoplus_{t \in [a]} A[t]\] and \[B = \bigoplus_{t \in [b]} B[t]\] where \([a] \times \omega \leq A[t]\] and \([b] \times \omega \leq B[t]\] for each \(t \in I\), and we may assume \([a] \leq [b] \times \omega\) for each \(t \in I\).

Let \(\Sigma\) be the collection of all ordered pairs \((a', n)\) where \(C \in I\), 

\[\{r^{a'}(t) : a' \in 2^{\omega} \}\] is an isomorphism from \(A[a']\) onto \(B[a']\), and \(\varepsilon\) is an isomorphism from \(A[\varepsilon]\) onto \(B[\varepsilon]\) extending \(\{r^{a'}(t) : a' \in 2^{\omega}\}\). Partially order \(\Sigma\) by defining \((a', n) \leq (b', m)\) if \(a' \subseteq b'\) and \((b', m) \leq (a', n)\) if \((a', n) \leq (b', m)\) and \((b', m) \leq (a', n)\) if \((b', m) \leq (a', n)\) and \((a', n) \leq (b', m)\). If \((a', n) \leq (b', m)\) is a chain in \(\Sigma\), define \(I = \bigcup_{a' \in 2^{\omega}} a'\) and define \(\varepsilon : A[I] \to B[I]\) by 

\[r(a') = \varepsilon(a') \text{ if } a' \in 2^{\omega}\]. Clearly \((a', \varepsilon) \in \Sigma\) and \((a', \varepsilon)\) is an upper bound for the chain \((a', n) : a' \in 2^{\omega}\). Thus we can apply Zorn's lemma to get a maximal element \((a', \varepsilon)\) of \(\Sigma\). We will assume \(I \neq \emptyset\) and arrive at a contradiction.

Let \(1 \in I\), we define by induction a collection \(\{K_n : n \in \omega \cup \{0\}\} \) as follows: Define \(K_0 = (I)\). Suppose \(n < \omega\) and \(K_n\) has been defined for each \(n' \leq n\).

Case 1, \(a = 0\)

Let \(K_a(0)\) be the minimum subset of \(I\) such that \(r^{a'}(A[\{a\}]) \subseteq B[K_0(0)]\) and \(r^{a'}(A[\{a\}]) \subseteq K_0(0)\). Let \(K_a(0)\) be the minimum subset of \(I\) such that \(r^{a'}(B[\{a\}]) \subseteq B[K_0(0)]\) and \(K_0(0) \subseteq K_a(0)\). Let \(K_a(0)\) be the maximum subset of \(I\) such that \(r^{a'}(B[\{a\}]) \subseteq B[K_a(0)]\) and \(K_a(0) \subseteq K_a(0)\). Continue in this
manner by induction, and define \( T' = \bigcup \lambda E \). 

Case 2: \( n \) even

Let \( K(n) \) be the minimum subset of \( I \) such that \( p^n(A[0]I_0) \subseteq A[0]K(n) \) and \( I_0 \subseteq K(n) \). Let \( K(n) \) be the minimum subset of \( I \) such that \( p^n(A[0]K(n)) \subseteq A[0]K(n) \) and \( K(n) \subseteq p^n(A[0]K(n)) \). Let \( K(n) \) be the minimum subset of \( I \) such that \( p^n(A[0]K(n)) \subseteq A[0]K(n) \) and \( K(n) \subseteq K(n) \). Continue in this manner by induction, and define \( T' = \bigcup \lambda K(n) \).

Define \( J = \bigcup \lambda / \) \( \lambda \subseteq I \) by construction, and \( (f_{p^n}(J)) \) is an isomorphism from \( p^n(J) \) onto \( p^n(J) \). Thus \( (f_{p^n}(J)) \) is an isomorphism from \( p^n(J) \) onto \( p^n(J) \). Let \( L = (J \cup J)') \). Then \( J \cup J = J \cup J \) and \( J \cup L \neq \emptyset \) since \( L \in L \).

Now \( A[J \cup J] = A(J) \otimes A(J) \), so \( p^n(J \cup J) = r(p^n(J \cup J)) \). Since \( A[J \cup J] = A(J) \) is totally projective and \( r(p^n(J \cup J)) \) is a complemented summand of \( p^n(J \cup J) \), it is clear from the proof of Prop 5 in [3] that we can choose a complemented summand \( M \) of \( s(J) \) in \( s(J \cup J) \) such that \( r(p^n(J \cup J)) \subseteq M \). Thus \( r(p^n(J)) = p^nM \) since \( p^n(J \cup J) = p^n(J \cup J) \otimes p^n(J \cup J) \). Let \( r(p^n(J \cup J)) \subseteq p^n(J \cup J) \).

Now \( B[J \cup J] = B(J) \otimes B(J) \), so \( A(J) \otimes A(J) \subseteq M \), and \( r(p^n(J \cup J)) \) is an isomorphism from \( p^n(J \cup J) \) onto \( p^n(J \cup J) \), since \( A(J) \subseteq M \), we can extend \( r(p^n(J \cup J)) \) to an isomorphism \( \tilde{r} \) from \( A(J) \) onto \( M \).

Now define \( \tilde{S} : A[J \cup J] \rightarrow B(J \cup J) \) by \( \tilde{S}(x) = g(x)xM \) if \( x \in A(J) \) and \( y \in A(J) \). Then \( \tilde{S} \) is an isomorphism from \( A[J \cup J] \) onto \( B(J \cup J) \), the pair \((J \cup J)\cup (J \cup J)\) is a member of \( S \), and \( (J \cup J) \cup (J \cup J) \), contradicting the maximality of \( (J \cup J) \). Thus \( J = \emptyset \), so \( T \) extends to an isomorphism from \( A[J] \) onto \( B(J) \). This completes the case where \( n = 0 \).

The only case remaining is that where \( n > 0 \). We may assume \( A = 0 \), since otherwise there is nothing to prove. Then \( A = 0 \) and \( B = 0 \) is a limit, so \( (p^n)(p^n(A)) = p^n(p^n(A)) \), \((p^n)(p^n(A)) = p^n(p^n(A)) \) and \( (p^n)(p^n(B)) = (p^n)(p^n(B)) \). Thus we can apply the induction hypothesis to extend \( f \) to an isomorphism \( T \) from \( p^n(A) \) onto \( p^n(B) \). Then, as shown above, \( T \) can be extended to an isomorphism from \( A \) onto \( B \). This completes the proof of Theorem 2.

**Conclusion:** If \( A \) is a totally projective group of length less than \( \aleph_0 \) and
a is a limit ordinal, then any automorphism of $\aleph^A$ can be extended to an automorphism of $A$.

This corollary is an extension of Theorem 3 in [5]. With a slight modification of the proof of Theorem 1, we can prove the following theorem, which, along with its corollary, generalizes Theorem 1 and Theorem 5 in [5].

**Theorem 3.** Let $\gamma$ be a limit ordinal less than $\omega_0$ and suppose that $A$ is a primary group such that $A\aleph^A$ is totally projective, then a primary group $B$ is isomorphic to $A$ if and only if $A\aleph^B$ is isomorphic to $\aleph^B$. Indeed, any isomorphism between $\aleph^A$ and $\aleph^B$ can be extended to an isomorphism between $A$ and $B$.

**Conclusion.** If $\gamma$ is a limit ordinal less than $\omega_0$ and $A$ is a primary group such that $A\aleph^A$ is totally projective, then every automorphism of $\aleph^A$ is induced by an automorphism of $A$.

**Chapter II: An Existence Theorem**

The purpose of this chapter is to prove an existence theorem analogous to those which accompany Ulam's Theorem and König's Theorem; that is, to describe those sequences of Ulam invariants which can be obtained from totally projective groups of length less than $\omega_0$. Since this theorem will be stated in terms of direct sums of countable groups, we will first state König's existence theorem.

Let $f$ be a cardinal-valued function on the countable ordinals. If there exists a countable ordinal $\alpha$ such that $f(\alpha) = 0$ for all countable ordinals $\alpha$, then the length of $f$ is the smallest such $\alpha$, otherwise the length of $f$ is $\omega$. The function $f$ of length $\lambda$ is admissible if whenever $\delta$ is an ordinal such that $\delta \times \lambda \leq \lambda$, there exists an infinite sequence $\{a_n\}$ of finite ordinals such that the $\delta(\kappa\alpha)$ is a set of countable ordinals which is admissible if the characteristic function is admissible. The symbol $H_n$ stands for the set of non-zero ordinals in the range of $f$, and $H(f)$ is the set of countable ordinals $\alpha$ such that $\alpha(f) = 0$. If $\aleph$ is an infinite cardinal, $D(\aleph)$ is the set of ordinals $\alpha$ in $H(f)$ for which $\alpha(f) \geq \aleph$. We define $D(\aleph)$ and $D(\aleph)$ as follows:

\[ H(f) = \{ \alpha \in D(\aleph) : 0 \leq \alpha(f) \} \]
\[ \aleph(\alpha) = \sup \{ \beta \in D(\aleph) : \beta < \alpha \} \]
\[ \aleph(\alpha) = \sup \{ \beta \in D(\aleph) : \beta < \alpha \} \]

**Theorem 4.** Let $f$ be an admissible cardinal-valued function on the
An extension of the Ulm-Keilen theorem

countable ordinals, then \( n \) is the Ulm function of a direct sum of countable reduced primary Abelian groups if and only if:

1. For each countable ordinal \( \kappa \) in \( \omega^* \), the set \( B(\kappa) \) is admissible, and
2. If the length of \( \tau \) is \( \kappa \), then \( \sup \{ f(\tau)_\alpha : \alpha < \kappa \} = \kappa \).

We will now set forth the terminology to be used in our existence theorem.

A proper function is a cardinal-valued function whose domain is the set of ordinals less than \( \omega \). If there is an ordinal \( \alpha < \omega \) such that \( f(\alpha) = 0 \) for all \( \beta \geq \alpha \), then the length \( \lambda(\tau) \) of \( \tau \) is the smallest such \( \lambda \), otherwise the length \( \lambda(\tau) \) is \( \omega \). If \( \tau \) is a proper function of length less than \( \omega \), then for each non-negative integer \( n \) such that \( \lambda(n) \leq \lambda(\tau) \), we define the \( n \)-th partial function \( f_n \) of \( \tau \) as follows:

\[
f_n(a) = f(\lambda(n) + a) \quad \text{for each} \quad a < n.
\]

We are now ready to state our existence theorem. Several technical lemmas will be presented before the proof of the theorem is given.

**Theorem 3.** A proper function \( \tau \) of length less than \( \omega \) is the Ulm function of a totally projective group if and only if:

1. For each non-negative integer \( n \) such that \( \lambda(n) \leq \lambda(\tau) \), the \( n \)-th partial function \( f_n \) of \( \tau \) is the Ulm function of a group \( G_n \) which is a direct sum of countable reduced primary groups.
2. \( \lambda(n) \leq \lambda^*(G_n) \) for each non-negative integer \( n \) such that \( \lambda(n) \leq \lambda(\tau) \).

**Lemma 1.** If \( G \) is a countable reduced primary group, then \( G \) can be written as a direct sum \( G = \bigoplus_{i \in I} \bf{K}_i \), where for each \( i \in I \) there is an ordinal \( \lambda_i \) such that \( \bf{K}_i \cong \bf{Z}(p) \), the cyclic group of order \( p \).

**Proof.** We induct on \( \lambda(G) \). If \( \lambda(G) = 1 \), the conclusion is obvious. Suppose \( \lambda(G) = \lambda > 1 \).

**Case 1.** \( \lambda = \omega \).

Then \( G \cong \bigoplus_{i \in I} \bf{Z}(p) \), where \( \bf{Z}(p) \cong \bf{Z}(\sqrt{p}) \) for each \( i \in I \). Thus \( [2, \text{Theorem } 1] \).

**Case 2.** \( \lambda \) is a limit ordinal.

Then \( [5, \text{Exercise } 3] \) \( G \) is a direct sum of groups of length less than \( \lambda \).
and we are finished by induction.

**Lemma 2.** If $G$ is a countable reduced primary group and $\lambda$ is an ordinal such that $\omega^\lambda \geq \lambda$, then for any $p \in P$ there is a countable reduced primary group $H$ such that $H^\lambda \mathcal{H}^\lambda \leq G$ and $\mathcal{H}^\lambda \mathcal{H}^\lambda \leq \lambda^\lambda \mathcal{H}^\lambda$.

**Proof.** Using Zippin's existence theorem [13], let $H$ be a countable reduced primary group of length $\omega^\lambda$ satisfying:

- $f_p(\alpha) = f_p(\alpha)$ for all $\alpha < \lambda$;
- $f_p(1) = 1$;
- $f_p(1) = 0$ if $1 < \alpha < \lambda$.

Consider the group $H^\lambda \mathcal{H}^\lambda$, it is routine to verify that this group has the same Ulm invariant as $G$ for all $\alpha < \lambda$. The $\mathcal{H}^\lambda$ Ulm invariant of $Y^\lambda \mathcal{H}^\lambda$ is $\text{diag}(\pi(Y^\lambda \mathcal{H}^\lambda)) = \pi(Y^\lambda \mathcal{H}^\lambda) = \lambda^\lambda \mathcal{H}^\lambda$.

Since it is clear that $Y^\lambda \mathcal{H}^\lambda \subseteq \lambda^\lambda \mathcal{H}^\lambda$, so $H^\lambda \mathcal{H}^\lambda \subseteq G$ by Zippin's theorem.

**Lemma 3.** If $G$ is a direct sum of countable reduced primary groups and $K$ is a reduced primary group such that $|K| = \text{sg}(G^\lambda)$, then there is a reduced primary group $\mathcal{P}$ such that $\mathcal{P}^\lambda \mathcal{P}^\lambda \leq K$ and $\mathcal{P}^\lambda \mathcal{P}^\lambda \leq G$.

**Proof.** Let $H = \bigoplus_{\alpha \in \mathcal{K}}$, where, for each $\alpha \in \mathcal{K}$, $\mathcal{K}$ is countable and for some ordinal $\mathcal{K}$, $\mathcal{K}^\alpha \leq \text{sg}(|G^\lambda|)$. Since $\lambda(0) = 0$, we have $\alpha \geq \mathcal{K}$, Lemma [7], we will construct a collection $\left\{ \mathcal{P}_{\alpha} : \alpha \in \mathcal{K} \right\}$ of pairwise disjoint subsets of $I$ such that $|\mathcal{P}_{\alpha}| = |\mathcal{P}|$ and such that $\lambda(\alpha(3)) = 0$. For each $\alpha \in \mathcal{K}$.

**Case 1.** $|\mathcal{P}_{\alpha}| = \mathcal{K}$.

Let $\left\{ \alpha_n : n < \mathcal{K} \right\}$ be a set of elements of $\mathcal{K}$ such that if $\omega^\lambda \geq \mathcal{K}$ and such that $\alpha_n \alpha^\lambda \leq \mathcal{K}$ for each $n < \mathcal{K}$. Let $\mathcal{P}_{\alpha} = \alpha_n \alpha^\lambda \leq \mathcal{K}$, we can partition $\mathcal{K}$ into a collection $\left\{ \mathcal{P}_{\alpha} : \alpha \in \mathcal{K} \right\}$, where $|\mathcal{P}_{\alpha}| \leq |\mathcal{P}_{\alpha}|$ for each $\alpha \in \mathcal{K}$. Then the collection $\left\{ \mathcal{P}_{\alpha} : \alpha \in \mathcal{K} \right\}$ is pairwise disjoint, $|\mathcal{P}_{\alpha}| = |\mathcal{P}_{\alpha}|$, and for each $\alpha \in \mathcal{K}$, $\lambda(\alpha(3)) = 0$.

**Case 2.** $|\mathcal{P}_{\alpha}| > \mathcal{K}$.

Let $\wedge$ be the first ordinal such that $|\alpha| = \mathcal{P}$, we construct by induction a collection $\left\{ \alpha_n : n < \mathcal{K} \right\}$ of pairwise disjoint subsets of $\mathcal{K}$ such that:

1. $|\alpha_n| = |\alpha_n|$ for each $n < \mathcal{K}$;
2. $\lambda(\alpha(3)) = 0$ for each $\alpha < \wedge$.
Define $I_0 = \{ i_0 : a < 0 \}$, a set of elements of $I$ indexed by the set of ordinals less than 0 such that if $a \neq b$ then $i_a \neq i_b$ and such that $\lambda(I_0) \geq a$ for each $a < 0$.

Now suppose $0 < u < \alpha$ and a collection $\{ I_1 : \lambda < \omega \}$ of pairwise disjoint subsets of $I$ satisfying (1) and (2) has been defined. Define $I' = \bigcup_{\lambda < \omega} I_\lambda$ and $I'' = \bigcap_{\lambda < \omega} I_\lambda$. Then $|I''| \leq \aleph_1$, $|\lambda| \leq |\lambda| \leq \omega$, so $\lambda(I''[\alpha]) = \alpha$. Thus we can let $I'' = \{ i : a < 0 \}$, a set of elements of $I''$ such that $\lambda(I''[a]) \geq a$ for each $a < 0$.

If we let $H = \{ u : u < \alpha \}$, then the collection $\{ I_1 : \lambda < \omega \}$ is pairwise disjoint, $|H| = |\lambda| = |\lambda|$, and for each $u < \alpha$, $\lambda(I''[u]) = \alpha$.

Thus in either case we have a collection $\{ I_1 : \lambda < \omega \}$ of pairwise disjoint subsets of $I$ such that $|H| = |\lambda|$, and such that $\lambda(I''[\alpha]) = \alpha$ for each $u < \alpha$.

We can assume without loss of generality that $\bigcup_{\lambda < \omega} I_\lambda = I$. Let $0$ be a one to one set function from $K$ onto $H$.

Suppose $k \in K$ and $(k) = k^D$. For each $i \in I(k)$, let $I_k$ be a countable reduced primary group such that $I_k/I_k^D I_k \geq I_k$ and $p^D I_k^D I_k = I_k \geq k^D$.

Define $A = K[I_k^D I_k]$. For each $k \in K$, define $C_k = 1_{I_k^D I_k}$, and let $C = \sum_{k \in K} C_k$. Define $G = A/C$. We note the following facts:

(a) $K \cap C = 0$.
(b) $(K \cap C)/C \cong p^D$.

Let $k \in K$ and let $a$ be a countable ordinal. Let $i \in I(k)$ such that $x_1 = 0$. Then $x_1 \in p^D I_k^D I_k \subset p^D I_k \subset p^D$, so $k^C = x_1 x_1 \in (p^D A/C)/C \subset p^D(A/C)$.

(c) $k^C = k[I_k^D I_k^D] = 1_{I_k^D I_k^D}$. 

(d) $O(k^C)/C \cong H$.

\[ O(k[I_k^D I_k^D]) = (A/C)/[K(k^C)/C] \cong A/(K(k^C)) = (K[I_k^D I_k])/(K[I_k^D I_k]) = \left( \left( I_k/2x_1 \right)/\left( I_k/2x_1 \right) \right) = \left( \left( I_k/2x_1 \right)/\left( I_k/2x_1 \right) \right) = \left( I_k/2x_1 \right)/\left( I_k/2x_1 \right) = I_k/2x_1 \cong I_k^D I_k \cong I_k^D I_k \cong I_k^D I_k \subset H. \]

(e) $(K \cap C) \subseteq C$.

Since $O(k[I_k^D I_k^D]) \cong H$ has length $G$, $p^D \cap C \subseteq (K \cap C)/C$.

Thus $p^D \cap C \subseteq (K \cap C)/C$.

Since $O(k[I_k^D I_k^D]) \cong H$ has length $G$, $p^D \cap C \subseteq (K \cap C)/C$.

Thus $p^D \cap C \subseteq (K \cap C)/C$.

THEOREM 1. If $a$ is a non-negative integer and $0, \alpha, \beta, \ldots, \nu$ are direct sum of countable reduced primary groups such that $|\alpha(a)| = |\alpha(a)| = |\alpha(a)|$ for $0 \leq k < n$, then there is a totally projective group $G$ such that $p^D G/G(p^D) = 0$ for $0 \leq k < n$, $|G| = |G|$, and $p^D G/G(p^D) = 0$. 

PROOF. The proof is by induction on \( n \). If \( n = 0 \), let \( G = \mathbb{Q} \). Suppose \( n > 0 \) and the theorem holds for all \( k < n \). Then by induction hypothesis there is a totally projective group \( G \) such that \( \mathbb{P}_{n}^{|G|} \leq \mathbb{P}_{n+1} \) for \( 0 \leq k < n+1 \), \( |G| = |\mathbb{Q}| \), and \( \mathbb{P}_{n}^{|\mathbb{Q}|} = 0 \).

Thus \( |\mathbb{Q}|^{|G|} \leq \mathbb{P}_{n}^{|\mathbb{Q}|} \), so by Lemma 3 there exists a reduced primary group \( G' \) such that \( \mathbb{P}_{n}^{|G'|} \leq |\mathbb{Q}|^{|G'|} \) and \( \mathbb{P}_{n}^{|\mathbb{Q}|} \leq |\mathbb{Q}|^{|G'|} \). If \( G' \neq 0 \), then \( \mathbb{P}_{n}^{|G'|} \leq |\mathbb{Q}|^{|G'|} \). We observe that \( \mathbb{P}_{n}^{|G'|} \leq \mathbb{P}_{n}^{|\mathbb{Q}|} = 0 \), so \( G' \) is totally projective since \( \mathbb{P}_{n}^{|G'|} = 0 \) and \( \mathbb{P}_{n}^{|\mathbb{Q}|} = 0 \).

LEMMA 5. If \( G \) is a totally projective group of length less than \( \omega \), then \( |\mathbb{P}_{n}^{|G|}| \leq \mathbb{P}_{n}^{|\mathbb{Q}|} \).

PROOF. \( G \) is a direct summand of a direct sum of groups of cardinality at most \( \mathbb{P}_{n}^{|\mathbb{Q}|} \), so \( G \) is a direct sum of groups of cardinality at most \( \mathbb{P}_{n}^{|\mathbb{Q}|} \).

Thus \( G = \bigoplus_{i \in I} G_i \), where \( |G_i| \leq \mathbb{P}_{n}^{|\mathbb{Q}|} \) for each \( i \in I \). Let \( J = \{ i \in I : \mathbb{P}_{n}^{|G_i|} \neq 0 \} \). If \( J \neq \emptyset \), then \( \mathbb{P}_{n}^{|G|} = 0 \) and there is nothing to prove.

If \( J = \emptyset \), then \( \mathbb{P}_{n}^{|G|} = \bigoplus_{i \in I} |G_i| \), so \( |G_i| \leq |\mathbb{Q}| \leq |\mathbb{Q}| \), but it is clear that \( \mathbb{P}_{n}^{|\mathbb{Q}|} \leq |\mathbb{Q}| \leq |\mathbb{Q}| \), so \( \mathbb{P}_{n}^{|G|} \leq |\mathbb{Q}| \).

PROOF OF THEOREM 3. Suppose that \( f \) is the Ulm function of a totally projective group \( G \). Then if \( n \) is a non-negative integer such that \( n < |G| \), we have \( f_{n}^{|G|} \leq f_{n}^{|\mathbb{Q}|} \).

Thus \( f_{\omega}^{|G|} \leq f_{\omega}^{|\mathbb{Q}|} \), which is a direct sum of countable reduced primary groups. Let \( \mathbb{P}_{n}^{|G|} = \bigoplus_{i \in I} \mathbb{P}_{n}^{|G_i|} \).

If \( n \) is a non-negative integer such that \( \mathbb{P}_{n}^{|G|} \neq \mathbb{P}_{n}^{|\mathbb{Q}|} \), we need \( |G_i| \leq \mathbb{P}_{n}^{|\mathbb{Q}|} \). Since \( \mathbb{P}_{n}^{|G_i|} \) is a totally projective group of length less than \( \omega \), and since \( \mathbb{P}_{n}^{|\mathbb{Q}|} \), we have by Lemma 5 that \( |G_i| \leq \mathbb{P}_{n}^{|\mathbb{Q}|} \leq \mathbb{P}_{n}^{|G_i|} \).

Conversely, suppose \( (1) \) and \( (2) \) hold. Let \( n \) be the smallest non-negative integer such that \( \lambda(G) \geq \mathbb{P}_{n}^{|G|} \). This makes sense since \( |\mathbb{Q}| \leq \mathbb{P}_{\omega}^{|G|} \). Then for \( 0 \leq k < n \), \( \lambda(G) \) is the Ulm function of a direct sum of countable reduced primary groups \( G_k \), and \( |G_k| \leq \mathbb{P}_{n}^{|G_k|} \) for \( 0 \leq k < n \), so by Lemma 4 there is a totally projective group \( G' \) such that \( \mathbb{P}_{n}^{|G'|} \leq \mathbb{P}_{n}^{|G|} \) for \( 0 \leq k < n \) and \( \mathbb{P}_{n}^{|G'|} = 0 \). Let \( k < n \) and let \( k \) be such that...
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\text{If } a \leq b(k+1), \text{ then } a = b(k+1) \text{ with } b < a, \text{ so } f(a) = f(b(k+1)) = f_b(s) = f_{b,k}(s) = f_{b,k}(b(k+1)) = f_{b,k}(b) \cdot f_b(s) = f_{b,k}(b) = f_{b,k}(a). \text{ Thus } f \text{ is the Ulm function of } G. \text{ The proof of Theorem 3 is complete.}

\textbf{BIBLIOGRAPHY}


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