THE GROUPS $P_2$ (*) (**)

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1. Introduction.

Totally projective groups were defined by Nunke [10]. His definition is homological, and he uses heavy homological machinery developed in [9] and [10] to derive their basic properties. Hill gave an alternate definition of totally projective groups, and proved that Ulm's theorem holds for that class [8]. Crawley and Hales defined a $p$-group to be a $T$-group if it is generated by a set of elements subject only to relations of the form $p^nx = 0$ and $px = y$. They proved Ulm's theorem for reduced $T$-groups [2, 3], and using Hill's result, showed that the class of reduced $T$-groups coincides with the class of totally projective groups. Crawley and Hales also introduced the useful concept of a $T$-basis, and proved that a $p$-group has a $T$-basis if and only if it is totally projective. The purpose of this paper is to introduce a class of groups $P_2$, one for each ordinal $\beta$, and to use systematically this class of groups to develop the properties of totally projective groups. In particular, the equivalence of the various definitions of totally projective groups will be proved. No homological machinery is needed.

The groups $P_2$ themselves, some of their relations with totally projective groups, and various facts about $T$-bases have been known to the author for many years, and appear in Chapter III of [1]. Recently, in a curious turn of events, these groups have been quite useful in Richman's constructive (in the sense of Bishop) development of the theory of countable $p$-groups [11].

The standard notations and terminology of Abelian group theory will be used, as found in [5], for example.

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2. The groups \( P_\beta \).

In this section, we will define the groups \( P_\beta \) and derive their basic properties.

**DEFINITION 2.1:** Let \( \beta \) be any ordinal number, and let \( p \) be a prime. Then \( P_\beta \) is the group generated by the set of tuples \( \{\beta\beta_1\ldots\beta_n\beta\} \) in an ordinal number, \( \beta > \beta_1 > \ldots > \beta_n \), subject to the relations \( p\beta\beta_1\ldots\beta_n\beta = \beta\beta_1\ldots\beta_n \) and \( p\beta = 0 \).

That is, \( P_\beta = F_\beta/K_\beta \), where \( F_\beta \) is the free group on the set \( \{\beta\beta_1\ldots\beta_n\beta \} \), and \( K_\beta \) is the subgroup of \( F_\beta \) generated by the elements \( p\beta\beta_1\ldots\beta_n\beta = \beta\beta_1\ldots\beta_n \) and \( p\beta = 0 \). Of course, we will denote the element \( \beta\beta_1\ldots\beta_n\beta \) of \( P_\beta \) simply by \( \beta\beta_1\ldots\beta_n \).

**THEOREM 2.2:** \( P_\beta \) is a \( p \)-subgroup and the order of \( \beta \) is \( p \).

**Proof:** It is obvious that \( P_\beta \) is a \( p \)-group. Let \( x \in Z(p^n) \) with \( o(x) = p \) and with \( px_{n+1} = x \). There is a homomorphism \( f: F_\beta \to Z(p^n) \) given by \( f(\beta\beta_1\ldots\beta_n\beta) = x_{n+1} \). Since \( px_{n+1} = x \), and \( px = 0 \), the kernel of \( f \) contains \( K_\beta \). Thus \( f \) induces a homomorphism \( g: P_\beta = F_\beta/K_\beta \to Z(p^n) \).

and \( g(\beta) = x_1 \). Since \( o(x_1) = p \) and \( px = 0 \), it follows that \( o(\beta) = p \).

Most of the technical details needed about \( P_\beta \) are consequences of the following theorem.

**THEOREM 2.3:** Let \( \alpha \) be an ordinal, \( \beta < \beta_1 \), and let \( X_\alpha \) be the subgroup of \( P_\beta \) generated by the set of all elements \( \beta\beta_1\beta_2\ldots\beta_\alpha \) of the form \( \beta\beta_1\beta_2\ldots\beta_\alpha \). Then \( P_\beta \bigg/ X_\alpha \approx \bigcup_{\gamma<\alpha} Q_\gamma \), where \( Q_\gamma \) is the direct sum of copies of \( P_\gamma \), one copy for each \( \beta\beta_1\beta_2\ldots\beta_\gamma \) with \( \beta_\gamma > \alpha \).

**Proof:** If \( \alpha = 0 \), the theorem is trivial. Suppose \( \alpha \neq 0 \). Let \( \gamma < \alpha \), and let \( s = \beta\beta_1\beta_2\ldots\beta_\gamma \), where \( \beta_\gamma > \alpha \). Let \( F \) be the free group on \( \{\gamma_1\gamma_2\ldots\gamma_\gamma\} \), where \( \gamma_1 > \ldots > \gamma_\gamma \). There is a homomorphism \( G \to P_\gamma \) which takes \( \gamma_1\gamma_2\ldots\gamma_\gamma \) to \( \gamma_1\gamma_2\ldots\gamma_\gamma \). Let \( S_\gamma \) be the set of all \( \beta\beta_1\beta_2\ldots\beta_\gamma \) with \( \beta_\gamma > \alpha \). Then we have a homomorphism

\[
G_\gamma = \sum_{e\in\gamma} F_e \to \prod_{e\in\gamma} P_\gamma = Q_\gamma ,
\]

and thus a homomorphism

\[
G = \sum_{e\in\gamma} G_\gamma \to \sum_{e\in\gamma} Q_\gamma .
\]
Let $F$ be the free group on the set \{\(\beta_1, \ldots, \beta_n, \alpha\)\}. Taking $F$ to 0, we have a homomorphism

\[ f: F \circledast G \to \sum_{\gamma \in \gamma} Q_\gamma. \]

But \(F \circledast G\) is the free group on \{\(\beta_1, \ldots, \beta_n, \alpha \beta_1 > \beta_1 > \ldots > \beta_n\)\}; that is, \(F \circledast G = P_\beta\). Furthermore, \(f\) takes $K_\gamma$ to 0 and also takes $X_\gamma$ to 0. Thus \(f\) factors through $P_\beta/X_\alpha$, and we have a homomorphism

\[ f: P_\beta/X_\alpha \to \sum_{\gamma \in \gamma} Q_\gamma. \]

Now for each $\gamma \in S_\alpha$, there is a homomorphism

\[ P_\gamma \to P_\beta/X_\alpha: \gamma_1 \cdots \gamma_n \mapsto \gamma_1 \gamma_2 \cdots \gamma_n + X_\alpha. \]

This yields a homomorphism

\[ g: \sum_{\gamma \in \gamma} \sum_{\gamma \in \gamma} P_\gamma \to P_\gamma/X_\alpha. \]

The maps \(f\) and \(g\) are inverses of each other.

**Corollary 2.4:** \(P_\beta/Z\beta \approx \sum_{\alpha < \beta} P_\alpha\).

**Lemma 2.5:** \(p^{\beta + 1}P_\beta = 0\).

**Proof:** Induct on $\beta$. Clearly, \(pP_\alpha = 0\). By 2.4, \(p^\beta(P_\beta/Z\beta) \approx \sum_{\alpha < \beta} P_\alpha\), which is 0 by the induction hypothesis. Since \(pZ\beta = 0\), it follows that \(p^{\beta + 1}P_\beta = 0\).

**Theorem 2.6:** \(p^\alpha P_\beta = X_\alpha\) for all $\alpha < \beta$.

**Proof:** Induct on $\alpha$. Certainly \(p^\alpha P_\beta = P_\beta = X_\alpha\). Now \(p^{\alpha + 1}P_\beta = p(p^\alpha P_\beta) = pX_\alpha = X_{\alpha + 1}\). For a limit ordinal $\alpha$, \(p^\alpha P_\beta = \bigcap_{\gamma < \alpha} p^\gamma P_\beta = X_\alpha\). We need \(p^\alpha P_\beta \cap X_\alpha\). By 2.3, \(p^\alpha(P_\beta/X_\alpha) \approx \sum_{\gamma \in \gamma} Q_\gamma = 0\) since \(p^\alpha P_\beta = 0\) by 2.5. Hence \(p^\alpha P_\beta \cap X_\alpha\). Therefore \(p^\alpha P_\beta = X_\alpha\).

**Theorem 2.7:** The length of $P_\beta$ is $\beta + 1$.

**Proof:** Let \(\lambda(0)\) denote the length of $G$. Proceeding by induction, we have \(\lambda(P_\alpha) = 1\), and \(P_\alpha/Z\beta = P_\beta/p^\beta P_\beta \approx \sum_{\alpha < \beta} P_\alpha\), which has length $\sup\{\alpha + 1|\alpha < \beta\} = \beta$. Hence \(\lambda(P_\beta) = \beta + 1\).
For each ordinal $\beta$, the group $P_{\beta}$ has length $\beta + 1$. If $x$ is a limit ordinal, then $\Lambda(\sum_{\beta < x} P_{\beta}) = x$. Therefore, at this point we have constructed, in an entirely elementary manner, $p$-groups of arbitrary length.

**Theorem 2.8:** Let $x + y = \beta$. Then $p^x P_{\beta} \approx P_{\beta}$. 

**Proof:** By 2.6, $p^x P_{\beta} = X_x$, and by definition, $X_x$ is the subgroup generated by the set $\{\beta \in \beta \in X_x > \beta \in \ldots > \beta > x\}$, which is the same as the subgroup generated by the set $S = \{\beta \in \beta \in S_x \beta > \beta > \ldots > \beta > x\}$. Let $Y_x$ be the free group on $S$. Then $(Y_x + K_{\beta})/K_{\beta} = X_x$, and $Y_x \cap K_{\beta}$ is generated by the elements $p\beta \in \beta \in \beta \in \ldots \beta \in S_x$ and $\beta \in S_x$, where $\beta \in S_x > x$. Therefore, $X_x = p^x P_{\beta}$ is the group generated by the elements $\{\beta \in \beta \in \beta \in \beta > \beta > \ldots \beta > S_x > x\}$ subject to the relations $p\beta \in \beta \in \beta \in \ldots \beta \in S_x = \beta \in \beta \in \beta \in \ldots \beta \in S_x = 0$. With each such $\beta \in \beta \in \beta \in S_x$ associate $\gamma_1 \ldots \gamma_x$, where $x + y = \beta$. This yields an isomorphism between $X_x = p^x P_{\beta}$ and $P_{\beta}$.

The basic idea in the proof of 2.8 enables us to prove the following theorem.

**Theorem 2.9:** Let $G$ be a direct summand of a direct sum of $P_{\beta}$’s, let $H$ be any group, and let $x$ be any ordinal. Then any homomorphism $f: p^x G \rightarrow p^x H$ extends to one from $G$ to $H$.

**Proof:** It suffices to prove the theorem for $G = P_{\beta}$. So suppose we have a homomorphism

$f: p^x P_{\beta} \rightarrow p^x H$

For each $\beta \in \beta \in S_x$ and for each $x \in S_x$, let $f(\beta \in \beta \in S_x) = h_{\beta \in \beta \in S_x}$. For each $\beta \in \beta \in S_x$ and for each $x \in S_x$, let $h_{\beta \in \beta \in S_x}$ be an element of $p^x H$ such that $p^{\beta \in \beta \in S_x} = h_{\beta \in \beta \in S_x}$. For each $x \in S_x$, let $h_{\beta \in \beta \in \ldots \beta \in S_x} = h_{\beta \in \beta \in \ldots \beta \in S_x}$. In this manner we get a homomorphism $F_{\beta} \rightarrow H$, and its kernel contains $K_{\beta}$. This yields a homomorphism $P_{\beta} \rightarrow H$, and since as pointed out in the proof of 2.8, $p^x P_{\beta}$ is the group generated by $\{\beta \in \beta \in \beta \in S_x > x\}$ subject to the relations $p\beta \in \beta \in \beta \in \beta \in \beta \in \ldots \beta \in S_x = \beta \in \beta \in \beta \in \beta \in \beta \in \ldots \beta \in S_x = 0$, this homomorphism agrees with $f$ on $p^x P_{\beta}$. This completes the proof. There are some significant special cases of 2.9.
Corollary 2.10: Let $G$ be a summand of a direct sum of $P_i$'s, and let $g$ be an element of $G$ of height $\alpha$ and order $p$. Let $H$ be any group and let $h \in (p^\alpha H)[p]$. Then there is a homomorphism $f: G \to H$ such that $f(g) = h$.

Proof: In $p^\alpha G$, $g$ has height 0 and order $p$, so that $Zg$ is a summand of $p^\alpha G$. Thus there is a homomorphism $f: p^\alpha G \to p^\alpha H$ such that $f(g) = h$. The homomorphism extends to one from $G$ to $H$ by 2.9.

Corollary 2.11: Let $H$ be any group and let $h \in p^\alpha H[p]$. Then there is a homomorphism $f: P_\alpha \to H$ such that $f(h) = h$.

In [4], Crawley and Hales introduced $T$-bases. This concept is a most interesting one, and it should enjoy more use than it has so far. We prefer to call $T$-bases $p$-bases.

Definition 2.12: A $p$-basis of a group $G$ is a subset $B$ of $G$ such that

1) if $b \in B$ and $pb \neq 0$, then $pb \in B$, and

2) every element of $G$ can be written uniquely in the form $\sum b_i n_i$ with $0 < n_i < p$.

For example, every cyclic $p$-group has a $p$-basis. If $g$ is a generator of such a group of order $p^{n+1}$, then $\langle g, g^2, \ldots, g^p \rangle$ is a $p$-basis. Similarly, if $x_i \in Z(p^n)$ with $x_i x_i = p_i$ and with $p_i x_i = x_i$, then $\{x_i \mid i \geq 1\}$ is a $p$-basis of $Z(p^n)$. The set $\{1, p, p^2, \ldots\}$ is a $p$-basis of $Z$, while $\{p^i \mid i \in \mathbb{Z}\}$ is a $p$-basis for the subgroup of rational numbers whose denominators are powers of $p$.

If $B$ is a $p$-basis of $G$, it is an easy exercise to show that the height of an element $\sum b_i n_i$, $0 < n_i < p$, is the minimal height of the terms $n_i b_i$.

Theorem 2.13: The set $\{z_1, \ldots, z_\alpha, \ldots, z_\beta\}$ is a $p$-basis of $P_\alpha$.

Proof: (Condition 1) of 2.8 is certainly met, and it is easy to see that every element of $P$ can be written in the form 2). To get uniqueness, induct on $\beta$. Suppose $\sum M\beta_1 \beta_2 \ldots \beta_\beta \beta_\alpha$ with $0 < m_\beta \leq p$, $0 < n < p$, and $0 < m_\beta < p$. Under the isomorphism $P_\alpha \cong \sum P_\beta$, $\sum M\beta_1 \beta_2 \ldots \beta_\beta \beta_\alpha$ goes to $\sum M\beta_1 \beta_2 \ldots \beta_\beta \beta_\alpha$. By induction,
\[ n_{p^k, A_i} = n_{p^k, A_i}. \] Since \( \alpha(\beta) = p \), then \( n = m \), and the proof is complete.

It is fairly obvious that direct sums of groups with \( p \)-bases have \( p \)-bases. It is not true that a summand of a group with a \( p \)-basis has a \( p \)-basis [12]. However, a summand of a \( p \)-group with a \( p \)-basis has a \( p \)-basis, but the only proofs known to the author involve Ulm's theorem or something close to it [8]. It is conceivable that the fact that summands of \( p \)-groups with \( p \)-bases have \( p \)-bases is not as deep as Ulm's theorem, and that there is an elementary proof of the fact. It would be interesting to see one.

3. **Totally projective groups and the groups \( P_3 \).**

Totally projective groups were defined by Naka via homological means [10]. In this section, we will give Hill's alternate definition of them and use the groups \( P_3 \) to derive some of their properties. Throughout, \( p \) will be a fixed prime, and all references to height will be to \( p \)-height.

The following two definitions are due to Hill [8].

**Definition 2.1:** A subgroup \( N \) of a group \( G \) is nice if every coset \( g + N \) has an element of the same height as the coset.

Thus \( N \) is nice in \( G \) if \( p^r(G/N) = (p^rG + N)/N \) for all \( r \). This is also equivalent to every coset having a representative of maximum height [8].

**Definition 2.2:** A reduced \( p \)-group \( G \) is **totally projective** if it has a system \( N \) of nice subgroups such that

a) \( 0 \in N \);  
b) unions of chains of subgroups in \( N \) are in \( N \);  
c) if \( N \in N \) and \( N \subseteq A \subseteq G \) with \( A/N \) countable, then there is an \( M \in N \) such that \( A \subseteq M \) and \( M/A \) is countable.

Hill proved [8] that two totally projective groups are isomorphic if they have the same Ulm invariants. We will need to use this below. Recently a much simpler proof than Hill's has been given [14].

**Lemma 3.3:** Any reduced \( p \)-group with a \( p \)-basis is totally projective.

**Proof:** Let \( G \) be a \( p \)-group with a \( p \)-basis \( B \). Let \( N \) be the system of subgroups of \( G \) generated by subbases \( S \) of \( B \) such that \( px \in S \) if \( s \in S \).
and if $ps \neq 0$. Let $N \in \mathcal{N}$. For $g \in G$, $g + N = \sum_{n \in \mathcal{N}, b + N}$ $n_b + N$ with $0 < n < p$, and $\sum_{n \in \mathcal{N}, b + N}$ is an element in $g + N$ of maximum height.

**Theorem 3.4:** A group is totally projective if and only if it is a direct summand of a direct sum of copies of $P_{\beta}'$. 

**Proof:** Suppose $G$ is totally projective. By 3.3, direct sums of copies of $P_{\beta}'$'s are totally projective. The $\beta$-th Ulm invariant of $P_{\alpha}$ is 1. Therefore,

$$G \oplus \sum_{\alpha < \beta \omega} \sum_{n \in \mathcal{N}} P_{\alpha}$$

and

$$\sum_{\alpha < \beta \omega} \sum_{n \in \mathcal{N}} P_{\alpha}$$

have the same Ulm invariants, and by Ulm's theorem for totally projective groups, are isomorphic.

Now suppose $G$ is a summand of a direct sum of copies of $P_{\beta}'$'s. By 2.9, the set $\{\beta_1, \ldots, \beta_n \}$ is a $p$-basis for $P_{\beta}$, and the appropriate union of these sets will be a $p$-basis $B$ of a direct sum $P$ of copies of $P_{\beta}'$'s. The system $\mathcal{N}$ of subgroups of $P$ gotten from $B$ as in the proof of 3.3 is a system of nice subgroups of $P$ satisfying the conditions a) and c) of 3.2 and the condition that unions of arbitrary subgroups in $\mathcal{N}$ are in $\mathcal{N}$. This condition is stronger than $b_1$ of 3.2, and the class of groups satisfying a) and c) and this stronger condition are closed under the taking of summands [8]. Indeed, if $P = A \oplus C$, then $\mathcal{M} = \{8 \subseteq A, S \subseteq T \subseteq \mathcal{N}$ for some $T \subseteq C \}$ is the required system for $A$. Therefore $G$ is totally projective.

**Corollary 3.5:** A totally projective group of length $<\omega$ is a direct sum of cyclic groups. A totally projective group of length $\omega > \omega$ is a direct sum of groups of cardinal $<|x|$. 

**Proof:** The proof of 3.4 shows that a totally projective group of length $x$ is a direct summand of a direct sum of $P_{\alpha}$'s with $\beta < x$. If $\beta < \omega$, then $P_{\beta}$ is finite, whence a totally projective group of length $<\omega$ is a direct sum of cyclic groups. If $\beta > \omega$, then $|P_{\beta}| = |\beta|$. A summand of a direct sum of groups of cardinal $< \lambda$ is a direct sum of groups of cardinal $< \lambda$ by Theorem 4.2 in [13], and the corollary follows.

**Corollary 3.6:** A totally projective group of length $<\Omega$, the first uncountable ordinal, is a direct sum of countable groups.
PROOF: Such a group is a direct summand of a direct sum of $P_\beta$'s with each $\beta$ countable, and hence with each $P_\beta$ countable.

COROLLARY 3.7: The class of totally projective groups is the same as that class of groups satisfying $a)$ and $c)$ of 3.2 and the condition that arbitrary group unions of members of $\mathcal{N}$ are in $\mathcal{N}$.

COROLLARY 3.8: Direct summands of totally projective groups are totally projective.

It would be interesting to see an elementary direct proof of 3.8. Note that Ulm's theorem for totally projective groups was used to get 3.8.

COROLLARY 3.9: Let $\alpha$ be an ordinal, let $G$ be totally projective and let $H$ be any group. Then any homomorphism $f: p^\alpha G \to p^\alpha H$ extends to one from $G$ to $H$.

PROOF: The corollary follows from 3.4 and 2.9.

4. Total purity and the groups $P_\alpha$.

The original definition of a totally projective group, given by Nunke in [10], is a reduced $p$-group $G$ such that for all $\alpha$, $p^\alpha \text{Ext}(G/p^\alpha G, X) = 0$ for all groups $X$. He also showed that such groups are precisely those reduced $p$-groups which are projective relative to a certain proper class of short exact sequences. This section is concerned with this relative homological aspect of totally projective groups. The following definition is Nunke's [10].

DEFINITION 4.1: A short exact sequence

$$0 \to A \to B \to C \to 0$$

is totally pure if the induced sequence

$$(p^\alpha B)[p] \to (p^\alpha C)[p] \to 0$$

is exact for all ordinals $\alpha$.

THEOREM 4.2: A short exact sequence $0 \to A \to B \to C \to 0$ is totally pure if and only if for every homomorphism $f: P_\alpha \to C$ there is a homomorphism $h: P_\alpha \to B$ such that $f = g \circ h$. 

PROOF: Suppose the condition on homomorphisms $P_\beta \to C$ is met.
Let $e \in (p^\alpha C)[p]$. Then there is a homomorphism $f: P_\alpha \to C$ such that $f(e) = c$. There is a homomorphism $h: P_\alpha \to B$ such that $g \circ h = f$. Then $h(e)$ is an element of $(p^\beta B)[p]$ such that $g(h(e)) = c$. Therefore the sequence is totally pure.

Now suppose $0 \to A \to B \to C \to 0$ is totally pure and that $f: P_\alpha \to C$ is a homomorphism. We induct on $|\beta|$, and if $|\beta| = 0$, then $P_\alpha$ is cyclic of order $p$ and the desired $h$ is given by $h(\beta) = b$, where $b \in (p^\beta B)[p]$ and $g(b) = f(b)$.

Now suppose our required $h$ exists for all $\alpha < \beta$. Let $f(\beta) = c$, and let $bc \in (p^\beta B)[p]$ such that $g(b) = c$. There is a homomorphism $h': P_\beta \to B$ such that $h'(\beta) = b$. The kernel of $f - g \circ h'$ contains $\beta$, so

$$f - g \circ h' = \lambda \cdot \eta,$$

where $\eta$ is the natural map $P_\beta \to P_\beta[Z_\beta]$. Since $P_\beta[Z_\beta] \cong \sum P_\alpha$, the induction hypothesis gives us a map $j: P_\beta[Z_\beta] \to B$ such that $g \cdot j = k$.

Let $h = h' + j \cdot \eta$. Then $g \circ h = g \circ (h' + j \cdot \eta) = g \circ h' + g \circ j \cdot \eta = f - k \circ \eta + k \circ \eta = f$, and the proof is complete.

**COROLLARY 4.3**: The class of totally pure short exact sequences is a proper class (in the sense of MacLane).

**PROOF**: If $S$ is any class of groups, then the class of short exact sequences $0 \to A \to B \to C \to 0$ such that for $S \in S$, each homomorphism $S \to C$ lifts through $B$ is well known to be a proper class.

Of course, one can prove directly that the totally pure sequences form a proper class, but it is a bit tedious.

With any proper class are the attendant relative Ext functors, long exact sequences, relative projectives and injectives, and relative projective and injective resolutions (if there are enough projectives
and injective). There seems to be very little known about the relative Ext functors for total purity, although some information is given in [20]. We will use our $P_r$'s to characterize the relative projectives and to get relative projective resolutions. The relative projective groups will simply be called projective.

**Lemma 4.4**: Divisible $p$-groups are projective.

**Proof**: It is enough to show that $Z(p^n)$ is projective. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be totally pure, and let $f: Z(p^n) \rightarrow C$. If $f \neq 0$, then $\operatorname{Im} f \cong Z(p^n)$, and we may as well assume that $C = Z(p^n)$. Let $x$ be an ordinal such that $p^x B$ is $p$-divisible. Let $c$ be an element of order $p$ in $C$. There is an element $b \in p^x B$ of order $p$ such that $g(b) = c$. Hence $p^x B$ contains a $Z(p^n)$ with socle $Zb$, and $g[Z(p^n)] = C$. It follows readily that $0 \rightarrow B \rightarrow C \rightarrow 0$ splits, and hence that $f$ can be lifted through $B$.

**Theorem 4.5**: There are enough projectives. That is, if $G$ is any group, then there is a totally pure exact sequence

$$0 \rightarrow K \rightarrow P \rightarrow G \rightarrow 0$$

with $P$ projective.

**Proof**: Let $G = H \oplus D$, where $D$ is the $p$-component of the divisible subgroup of $G$. The $p$-primary component of $H$ is reduced. Let its length be $\lambda$. Let $F \rightarrow H$ be an epimorphism with $F$ free. Let $F = \bigoplus_{(p,M)} P_M$. We get an epimorphism

$$F \oplus D \oplus P \rightarrow H \oplus D$$

and the attendant short exact sequence is totally pure by 4.2. The group $F \oplus D \oplus P$ is projective by 4.2 and 4.4.

**Corollary 4.6**: A group is projective if and only if it is of the form $F \oplus D \oplus P$, where $F$ is free, $D$ is a divisible $p$-group, and $P$ is a direct summand of a direct sum of $P_r$'s. In particular, the reduced $p$-groups which are projective are precisely the totally projective groups.

**Proof**: Let $G$ be projective, and let $G = H \oplus D$, where $D$ is the $p$-component of the divisible part of $G$. By 4.5

$$H \oplus K = F \oplus P$$

where $F$ is free and $P$ is a direct sum of $P_r$'s. Project $H$ into $F$. The
image of this projection is free, so $H = L \oplus M$, where $L$ is free and $M \subseteq F$. Since $H$ is a summand of $F \oplus F$, $M$ is a summand of $F$, and so $G$ has the desired form. The other half of the proof is clear.

It would be interesting to characterize those groups of relative dimension 1. Such a group $G$ is one for which there is a totally pure exact sequence

$$0 \to K \to P \to G \to 0$$

with $P$ and $K$ projective. Note that $p$-groups $G$ with no elements of infinite height have dimension 1 since an ordinary pure resolution $0 \to K \to P \to G \to 0$ with $P$ a direct sum of cyclic groups is a totally pure sequence with $P$ and $K$ projective.

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BIBLIOGRAPHY