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HIGH EXTENSIONS OF ABELIAN GROUPS

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To professor L. Fuchs

By

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1. Introduction

High subgroups of Abelian groups were introduced by one of the authors [6], and have been investigated in several papers [6]; [7]; [8]; [9]; [10]. The importance of homological techniques in Abelian groups has prompted the authors to investigate the homological properties of high subgroups. This paper is concerned with such properties.

If $G$ is an Abelian group, and $H$ is a subgroup of $G$ maximal with respect to $H \cap G^1 = 0$, where $G^1 = \bigcap nG$, then $H$ is called a high subgroup of $G$. Two fundamental properties of high subgroups that will be used here are: (a) $H$ is a high subgroup of $G$, then (a) $H$ is pure in $G$ and (b) $G/H$ is divisible. A proof of these facts is in [8].

DEFINITION 1. If $H^1 = 0$ and $D$ is divisible, then the exact sequence $0 \to H \to X \to D \to 0$ is called a high exact sequence or a high extension of $H$ by $D$ if $f(H)$ is a high subgroup of $X$.

For any two groups $A$ and $B$, $\text{Ext}(A, B)$ can be defined as equivalence classes of short exact sequences $0 \to B \to X \to A \to 0$ with addition being Baer addition ([2], p. 290). For our purposes it is useful to consider $\text{Ext}(A, B)$ in this way.

In Section 2, we will be concerned with two principal questions. If $H^1 = 0$ and $D$ is divisible, which elements in $\text{Ext}(D, H)$ are represented by high extensions $0 \to H \to X \to D \to 0$? What groups $H$ with $H^1 = 0$ are the high injectives, i.e. what groups $H$ with $H^1 = 0$ are summands of every group $G$ in which they are high? These questions are answered in Theorems 1, 2, and 3. In particular it turns out that the set $\text{HExt}(G, H)$ of high extensions is a subgroup of $\text{Ext}(G, H)$. This result suggests the following problem. Find a functor $\text{HExt}(A, B)$ such that when $A$ is divisible and $B^1 = 0$, $\text{HExt}(A, B) = \text{HExt}(A, B)$. Such a functor is exhibited in Section 3, and some of its properties are noted. The existence of this functor makes

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‡ All groups considered in this paper are Abelian.

* It is clear that any extension equivalent to a high extension is a high extension. We will abuse the language by referring to the elements of $\text{Ext}(A, B)$ as extensions rather than equivalence classes of extensions.
possible a more general definition of high subgroups. These more general high subgroups are characterized in Theorem 4.

In Section 4, a still more general, and perhaps more natural, definition of high subgroups is given. The results in Section 4 are completely analogous to the results in Section 3 (although they do not imply the results of Section 3), and yet the concepts involved are of such a nature that they can be dualized. These dual concepts will be the subject of a subsequent paper.

In Sections 2, 3 and 4 a subgroup $\operatorname{Shom}(A, B)$ of $\operatorname{Hom}(A, B)$ plays a central role. This subgroup consists of all $f \in \operatorname{Hom}(A, B)$ such that $ker\, f$ is an essential subgroup of $A$. $\operatorname{Shom}(A, B)$, considered as a functor of $B$, is left exact. Its right derived functor is related to $\operatorname{Ext}(A, B)$ and to high exact sequences, in the senses of both Sections 3 and 4. Section 5 is devoted to these functorial properties of $\operatorname{Shom}$.

2. High extensions

A central concept in our discussion is that of an essential extension. An extension

$$0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0$$

is essential if $F$ a subgroup of $X$ and $A \cap M = 0$ implies $M = 0$.

The subgroup $A$ is also called an essential subgroup of $X$. A homomorphism

$$f: X \rightarrow Y$$

is an essential homomorphism if $ker\, f$ is an essential subgroup of $X$.

**Lemma 1.** Let $\operatorname{Shom}(X, Y)$ be the set of essential homomorphisms in $\operatorname{Hom}(X, Y)$. Then $\operatorname{Shom}(X, Y)$ is a subgroup of $\operatorname{Hom}(X, Y)$.

**Proof.** Let $f, g \in \operatorname{Shom}(X, Y)$. Then $(ker\, f) \cap (ker\, g) \subseteq ker\, (f - g)$. Let $S$ be a subgroup of $X$. If $(ker\, f) \cap (ker\, g) \subseteq S = 0$, then $(ker\, f) \cap (ker\, g) \subseteq S = 0$. Hence $S = 0$. Thus $(ker\, f) \cap (ker\, g)$, and hence $ker\, (f - g)$, is essential in $X$. Thus $\operatorname{Shom}(X, Y)$ is a subgroup of $\operatorname{Hom}(X, Y)$.

It is well known [5] that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is pure exact, then

$$0 \rightarrow \operatorname{Hom}(X, A) \rightarrow \operatorname{Hom}(X, B) \rightarrow \operatorname{Hom}(X, C) \rightarrow \operatorname{Pext}(X, A) \rightarrow \operatorname{Pext}(X, B) \rightarrow \operatorname{Pext}(X, C) \rightarrow 0$$

and

$$0 \rightarrow \operatorname{Hom}(C, X) \rightarrow \operatorname{Hom}(B, X) \rightarrow \operatorname{Hom}(A, X) \rightarrow 0$$

$$0 \rightarrow \operatorname{Pext}(C, X) \rightarrow \operatorname{Pext}(B, X) \rightarrow \operatorname{Pext}(A, X) \rightarrow 0$$

are exact, where $\operatorname{Pext}(X, Y)$ is the set of pure exact sequences in $\operatorname{Ext}(X, Y)$. Also, any group $A$ is a pure subgroup of an algebraically compact group $C$. That is, there exists a pure injective resolution $0 \rightarrow A \rightarrow C \rightarrow A \rightarrow 0$ for any group $A$ ([1], p. 85). The following lemma is crucial.

**Lemma 2.** Let $0 \rightarrow H \rightarrow B \rightarrow H \rightarrow 0$ and $0 \rightarrow H \rightarrow C \rightarrow C \rightarrow 0$ be pure injective resolutions of $H$. Then $\operatorname{Shom}(A, B/H)$ and $\operatorname{Shom}(A, C/H)$ have the same image in $\operatorname{Pext}(A, H)$.

**Proof.** Since $H$ is a pure subgroup of $C$ and $B$ is pure injective there is a homomorphism $g: C \rightarrow B$ which extends the map $H \rightarrow B$. This induces maps $g': C/H \rightarrow B/H$ and $g'': \operatorname{Shom}(A, C/H) \rightarrow \operatorname{Shom}(A, B/H)$. Let $f' \in \operatorname{Shom}(A, C/H)$. $f'$ maps onto the
element of $\text{Pext}(A, H)$ represented by $0 \to H \to G \to A \to 0$ where $G = \{(c, a) : c \in C$, $a \in A, f(a) = c + H\}$. The homomorphism $\gamma : \text{Shom}(A, B/H)$ maps onto the element represented by $0 \to H \to F \to A \to 0$ where $F = \{(b, a) : b \in B, a \in A, g^*(a) = b + H\}$. Since $g^*(f)(a) = g^*(f)(a) = g(c + H) = g(c) + H$, we can define a homomorphism $\pi : G \to F$ by $\pi(c, a) = (g(c), a)$. Suppose $\pi(c, a) = 0$. Then $a = 0$ implies $c \in H$. But $g/H$ is the identity map, so $g(c) = 0$ implies $c = 0$. Hence $\pi$ is a monomorphism. Let $(b, a) \in F$. Then $f(a) = c + H \in C/H$, and $g^*(f)(a) = b + H = g(c) + H$ implies $b = g(c) + h$ for some $h \in H$. Thus $g(c) + h = g(c) + g(b) = c + b + H$ and since $f(a) = c + h + H$, then $(c + h, a) \in G$ and $\pi(c + h, a) = (b, a)$. Therefore $\pi$ is an isomorphism. It is easily checked that the sequences are equivalent, using the isomorphism $\pi$. This proves that the image of $\text{Shom}(A, B/H)$ is contained in the image of $\text{Shom}(A, C/H)$. By symmetry we have the desired equality.

The following theorem relates Shom to Hext.

**Theorem 1.** Let $H$ be a group with no elements of infinite height (i.e., $H^1 = 0$), and $D$ a divisible group. If $0 \to H \to C \to H \to 0$ is a pure injective resolution of $H$, then $\text{Shom}(D, C/H) \to \text{Hext}(D, H) = 0$ is exact. In particular, $\text{Hext}(D, H)$ is a subgroup of $\text{Pext}(D, H)$.

**Proof.** Let $g : \text{Shom}(D, C/H)$. Then $g$ maps onto the pure exact sequence $0 \to H \to G \to D \to 0$ where $G = \{(c, d) : c \in C, d \in D, g(d) = c + H\}$. Let $K = H = \{(0, d) : (0, d) \in G\}$. If $(0, d) \in K$ and $n$ is any positive integer, there is an $x \in D$ such that $nx = d$. Now $g(x) = c + H$ with $n(c + H) = g(d) = 0$. Since $H$ is pure in $C$, we can choose a representative $c$ such that $n = 0$. Then $(c, x) \in G$ and $n(c, x) = = (0, d) \in G$. Thus $K$ is contained in $G$.

Clearly $H^{n+1} \cap K = 0$. The quotient $(H + K)/H$ is an essential subgroup of $G/H = D$. Since $(H + K)/H = \ker g$, suppose $N$ is a subgroup of $G$ and $(N + H)/H = \ker g$. Then $((N + H)/H)/(K + H)/H = H$ implies $N + H = H$, and so $N \subseteq H$. Therefore $N = 0$. Thus $K < H$ is essential in $G$ and, since $H$ is pure in $G$, this implies that $H$ is maximal disjoint from $K$. If $H \cap L = 0$ then $L \cap K = 0$ and so $L = H$. Therefore $H$ is maximal disjoint from $G^1$, and the sequence $0 \to H \to G \to D \to 0$ is exact.

Now suppose $0 \to H \to G \to D \to 0$ is exact. Let $0 \to G^1 \to B$ be a pure injective resolution of $G^1$. Then $0 \to H^1 \to B \to B/(H^1)$ is a pure injective resolution of $H$, where $Z : H \to G^1$ is the natural map. Define $f : D \to G \to B/(H^1)$ by $f(g + h) = (g + G^1)/(H + G^1)$. Then $\ker f = (H + G^1)/H$ is an essential subgroup of $D$, so $\subseteq \text{Shom}(D, B/(H^1))$. By what has already been proved, $f$ maps onto a sequence $0 \to H \to X \to D \to 0$ in $\text{Hext}(D, H)$, where $X = \{(b, d) : b \in B, d \in D, f(d) = b + H\}$. Define $\gamma : G \to X$ by $\gamma(g) = (g + G^1, g + H)$. The map $\gamma$ is clearly a homomorphism. Let $(b, d) \in X$. Then $d = g + H$ and $b + a(H) = f(g + H) = g + G^1 + a(H)$. Thus $b = (g + G^1) + (b + G^1) = (g + h) + G^1$ for some $h \in H$, and $\gamma(g + h) = (g + h + G^1, g + h + H) = (b, g + H)$. Thus $\gamma$ is an isomorphism. It is easily checked that the diagram

$$
\begin{array}{c}
0 \to H \to G \to D \to 0 \\
\downarrow \left(\begin{array}{c}
\pi \\
\gamma
\end{array}\right) \\
0 \to H \to X \to D \to 0
\end{array}
$$
is commutative. Thus $f$ maps onto $0 \rightarrow H \rightarrow G \rightarrow D \rightarrow 0$, and $\text{Shom}(D, B/H)$ maps onto $\text{Hext}(D, H)$. By Lemma 2, this is the same as the image of $\text{Shom}(D, C/H)$.

When $D$ is torsion, $\text{Hext}(D, H)$ has a surprisingly simple description as a subgroup of $\text{Pext}(D, H)$. To achieve this description we need a few facts.

**Lemma 3.** If $A$ is torsion and $E$ is divisible then

$$\text{Shom}(A, E) = \bigcap_p \text{Hom}(A, E)$$

where $p$ is prime.

**Proof.** Let $f \in \text{Shom}(A, E)$, and $p$ be a prime. If $a, b \in A$ and $pa = pb$, then $f(a - b) = 0$. Thus we can define a homomorphism $h : A \rightarrow E$ by $h(pa) = f(a)$. Since $E$ is divisible, $h$ may be extended to a homomorphism $g : A \rightarrow E$. Clearly $pg = f$, so that $\text{Shom}(A, E) \subseteq \bigcap_p \text{Hom}(A, E)$. The other inclusion is trivial since $A$ is torsion.

Let $Q$ denote the additive group of rational numbers and $Z$ the additive group of integers. If $H^1 = 0$ there is a natural embedding of $H$ as a pure subgroup of $C = \text{Ext}(Q/Z, H)/\text{Pext}(Q/Z, H)$ [5]. The group $C$ is algebraically compact [4] and $C/H$ is divisible [5]. Let $D$ be a divisible group. Since $\text{Pext}(X, Y) = \text{Ext}(X, Y)_1$ ([3], p. 246), $C$ is reduced, and the pure exact sequence $0 \rightarrow H \rightarrow C \rightarrow C/H \rightarrow 0$ yields the exact sequence.

$$0 \rightarrow \text{Hom}(D, C/H) \rightarrow \text{Pext}(D, H) = 0.$$ 

Thus we have

**Theorem 2.** If $H^1 = 0$ and $D$ is torsion divisible then

$$\text{Hext}(D, H) = \bigcap_p \text{Pext}(D, H).$$

We conclude this section with a characterization of the high injectives.

**Theorem 3.** Let $H^1 = 0$. Then $H$ is high injective (i.e., $H$ is a summand of every group in which it is high) if and only if $\text{Pext}(Q/Z, H) = 0$.

**Proof.** Suppose $H^1 = 0$ and $\text{Pext}(Q/Z, H) = 0$. If $0 \rightarrow H \rightarrow G \rightarrow C/H \rightarrow 0$ is high exact, then

$$0 \rightarrow \text{Hom}(Q/Z, G) \rightarrow \text{Hom}(Q/Z, G/H) \rightarrow \text{Pext}(Q/Z, H) = 0 \rightarrow \text{Pext}(Q/Z, G) = 0.$$ 

is exact. Write $G = R \oplus D$, with $R$ reduced, $R \supset H$, and $D$ divisible. Then $\text{Pext}(Q/Z, R) \approx \text{Pext}(Q/Z, G) = 0$. Since there is a natural embedding of $R$ in $\text{Ext}(Q/Z, R)$ and $\text{Pext}(Q/Z, R) = \text{Ext}(Q/Z, R)_1$, it follows that $R^1 = 0$. But $H$ was high in $G$. Hence $H = R$, which is a summand of $G$.

Suppose now that $H^1 = 0$ and $\text{Pext}(Q/Z, H) = 0$. Then $\text{Ext}(Q/Z, H) = 0$. This last quotient group is torsion free divisible [5], and it has a subgroup $G/H = Q$ such that $G \cap \text{Pext}(Q/Z, H) = 0$. Since $G$ is pure in $\text{Ext}(Q/Z, H)$, $G^1 = 0$, and it is easy to see that $H$ is a high subgroup of $G$. However, $G$ is reduced, and so $H$ is not a summand of $G$. In other words, $H$ is not high injective.

The above theorem gives an explicit answer to the question — which groups $H$ with $H^1 = 0$ can be high subgroups of some reduced group $G$ with $G^1 = 0$. 

We remark that a group $H$ such that $H^t = 0$ is also high injective if and only if the torsion subgroup $H_t$ of $H$ is high injective. This follows from the exact sequence

$$0 \rightarrow \text{Pext} (\mathbb{Q}/\mathbb{Z}, H) \rightarrow \text{Pext} (\mathbb{Q}/\mathbb{Z}, H) \rightarrow \text{Pext} (\mathbb{Q}/\mathbb{Z}, H/H_t) = 0.$$ 

A torsion group $T$ is high injective if and only if each primary component $T_p$ of $T$ is closed (313, page 114). Thus the high injectives may be described as those groups $H$ such that $H^t = 0$ and each primary component of $H_t$ is closed. In particular, any reduced torsion free group is high injective.

### 3. Pure-high extensions

We will now extend the definition of high and, in particular, will exhibit a functor $\text{Hex}^d(B, A)$ such that if $B$ is divisible and $A' = 0$, then $\text{Hex}^d(B, A) = \text{Hex}^d(B, A')$.

If $0 \rightarrow A \rightarrow C \rightarrow A \rightarrow 0$ is any pure injective resolution of $A$, then we have from Lemma 2 that the image of $\text{Shom} (B, C)/A$ is a well defined subgroup of $\text{Pext} (B, A)$. This makes possible the following

**Definition 2.** Let $A$ and $B$ be any groups, and let $0 \rightarrow A \rightarrow C \rightarrow A \rightarrow 0$ be a pure injective resolution of $A$, $A$ sequence $0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0$ is a pure-high exact sequence, and $A$ is a pure-high subgroup of $X$, if and only if it is in the image of the map $\text{Shom} (B, C)/A$. The image of this map is denoted by $\text{Hex}^d(B, A)$.

It follows immediately that a high subgroup of a group is a pure-high subgroup of that group, and that if $A$ is pure-high in $X$, $A' = 0$, and $X/A'$ is divisible, then $A$ is a high subgroup of $X$. Furthermore, if $B$ is divisible and $A' = 0$, then $\text{Hex}^d(B, A) = \text{Hex}^d(B, A')$.

The elements of $\text{Hex}^d(B, A)$, that is, the pure-high sequences, can be described in a manner analogous to the definition of high sequences. This description gives considerably more insight into the concept of pure-high, and it is contained in

**Theorem 1.** The exact sequence $0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0$ is a pure-high extension if and only if there exists a subgroup $K$ of $G$ such that $A$ is maximal disjoint from $K$ and $(A + K)/K$ is pure in $G/K$.

**Proof.** Suppose $0 \rightarrow A \rightarrow G \rightarrow B \rightarrow 0$ is a pure-high extension of $A$ by $B$. Let $0 \rightarrow A \rightarrow C \rightarrow A \rightarrow 0$ be a pure injective resolution of $A$. Then there is a $g$ in $\text{Shom} (B, C)/A$ with $G \cong \{(c, b) | c \in C, b \in B, g(b) = c + A\}$. Let $K$ be the subgroup of $G$ corresponding to $K' = \{(a, k) | a \in A, g(k) = 0\}$.

From the proof of Theorem 1, $A$ is maximal disjoint from $K$. Let $a + K \in (A + K)/K$. Then $a + K = n(c + K)$ and there is a corresponding equation $(a, 0) + K' = n(c, b) + K'$, so that $(a, 0) = n(c, b) + (0, K)$ for some $k \in B$ with $g(k) = 0$. Now $a = nc$ and, since $A$ is pure in $C$, $a = na'$ for some $a' \in A$. Thus $a + K = n(a' + K) + n(A + K)/K$, and $(A + K)/K$ is pure in $G/K$.

Substituting $n'$ for $G'$ the proof of the converse is the same as in Theorem 1.

It is immediate from Theorem 4 that $A$ is a pure-high subgroup of $G$ if and only if there is a subgroup $K$ of $G$ such that $A \cap K = 0$, $A + K$ is essential in $G$ and
Theorem 5. Let $A$ and $K$ be subgroups of $G$. The following hold.

(a) If $A$ is maximal disjoint from $K$ then $(A + K)/K$ is pure in $G/K$ (and hence $A$ is pure-high with respect to $K$) if and only if $K \cap (A + nG) = K \cap nG$ for all $n$.

(b) If $K \subseteq G$, and $A$ is maximal disjoint from $K$, then $A$ is pure-high with respect to $K$, and in particular $A$ is pure in $G$.

(c) If $A$ is maximal disjoint from $K$ and $G/A$ is divisible, then $A$ is pure-high with respect to $K$ if and only if $K \subseteq G$.

(d) If $A$ is pure-high with respect to $K$ then, in $G/A$, $A$ is pure-high with respect to $K_A$.

(e) If $A$, is pure-high with respect to $K$, in $G/A$, then $A$ is pure-high with respect to $\Sigma K_A$ in $G_A$.

(f) If $A$ is pure-high in $G$ and $A \subseteq K < G$, then $A$ is pure-high in $K$.

(g) If $A$ is pure-high in $G$ and $K \subseteq A$, then $A/K$ is pure-high in $G/K$.

(h) If $A$ is pure-high in $G$ and $K$ is pure-high in $G$, then $A$ is pure-high in $G$.

(i) If $A$ is pure-high in $G$ and $A/K$ is pure-high in $G/K$, then $A$ is pure-high in $G$.

Proof. (a) Let $A$ be maximal disjoint from $K$ and assume $K \cap (A + nG) = K \cap nG$ for all integers $n$. Suppose $A$ is not pure in $G$, and let $n$ be the smallest positive integer such that $A \cap nG \neq nA$. Let $nx \in A \setminus nA$, and write $n = pr$, $p$ a prime. Since $A$ is maximal disjoint from some subgroup of $G$, $A$ is neat in $G$, and there is an $a \in A$ such that $nx = pxr = pr$. By the choice of $n$, $r \not\in A$, and hence $-a = axr \in A$. Thus there is $b \in A$ and an integer $m$ such that $(m, p) = 1$ and $0 \neq m (px - a) + b \in K$. Let $s$ and $t$ be integers such that $sm + tp = 1$. Then $s(px - a) = r$, and $a = sx - a + sbK \cap (A + nG) = K \cap nG$. Now $r(px - a) + b = r + b < (A + nG)K = K \cap nG$. Since $r = -a$, $s = a$, and $t = (y - x) \in K \cap nG = A$, so $b = a = re$ for some $c \in A$. This contradiction proves that $A$ is pure in $G$.

Let $a + K = (A + K)/K \cap (G/K)$. Then $a = mg + k$ with $g \in G$, $k \in K$, and $k \in K \cap (A + nG) = K \cap nG$. So that $a \in K \cap nG = nA$, and $a + K = n(A + K) = nA$. Therefore $A$ is pure-high with respect to $K$.

The converse is straightforward, and (b) and (c) are immediate corollaries of (a). We remark that (b) gives a generalization of the fact that high subgroups are pure.

(e) Let $A_i$ be pure-high with respect to $K_i$ in $G_i$, $i = 1, 2$, and let $A = A_1 \oplus A_2$. Clearly $(A + K)/K$ is pure in $G/K$. Suppose $A$ is not maximal disjoint from $K$ in $G$. Let $B$ be a subgroup of $G$ containing $A$, which is maximal disjoint from $K$, and let $B_i$ be the projection of $B$ into $G_i$, $i = 1, 2$. Now $A_i \subseteq B \cap G_i < B_i$, and since $(B \cap G_i) \cap K_i < B_i \cap K_i = 0$, then $B \cap G_i = A_i$. If $B_1 = A_1$, then $B_2 < B$ so that $B_2 = B \cap G_2 < A_2$ and $B = A_1 \oplus A_2$. Thus $A_1 \neq A_2$ and $B \cap K_i \neq 0$. Let $b_j \in B_j \cap K_j$, $j \neq 0$. There is $b_2 \in B_2$ such that $b_2 = b_1 \in B$, and clearly $b_2 \in A_2$. For some $a_1 : A_2$ and some integers $n, b_1, a_2 = k_j \in K_j$. Now $a_1 : b_2 + b_1 = k_j = k_j \neq 0$. This contradiction proves that $A$ is maximal disjoint from $K$ in $G$, and therefore $A$ is pure-high in $G$. The general case now follows easily.
(c) Suppose $A$ is pure-high with respect to $B$ in $K$ and $K$ is pure-high with respect to $L$ in $G$. Let $H$ be a subgroup of $G$ containing $A$, and suppose $H \cap (B + L) = 0$. If $H \subseteq K$ then $(H \cap K) \cap L = 0$, and $h + k = x + 0$ with $h \in H$, $k \in K$, and $x \in L$. Since $H \cap L = 0$ and $A \subseteq H$, then $k \in A$. Thus $nk + a = b + c$ with $a \in A$, $b \in b$, and $c = x$. Then $nk + a = b + c$ with $k \in K$, $a \in A$, $b \in b$, and $c = x$. Therefore $H \subseteq K$, and $A \subseteq H$. $H \cap B = 0$ implies $H = A$. Thus $A$ is maximal disjoint from $B + L$.

Let $a + (B + L) \subseteq (A + (B + L))(B + L) = \cap (G((B + L))$. If $a + (B + L) = ng + (B + L)$ then for some $b \in B$, $a + b + L = ng + L \subseteq (K \cap L) = (G((B + L))$. Thus $a \cap L = n + k$, $k \in K$, and since $L \cap K = 0$, $a + b = nk$. Now $a + B = nk + B \subseteq (A + B)/(B) \cap (K(B)) = \cap (A + B)/(B)$, so that $a + nA$ and $a + (B + L) \subseteq (A + B)/(B)$.

Therefore $A$ is pure-high (with respect to $B + L$) in $G$.

(i) Suppose $K$ is pure-high with respect to $N$ in $G$ and $A$ is pure-high with respect to $M$ in $G$. Let $L = N \cap M$. Then $A \cap L = (A \cap M) \cap N = K \cap N = 0$. Suppose $A \subseteq B$ with $B \cap L = 0$. Then $K \subseteq B \cap M$ and $(B \cap M) \cap N = 0$ implies $B \cap M = K$, so that $(A \subseteq B)/K$ and $(B/K) \cap (M/K) = K$ implies $B = A$. Thus $A$ is maximal disjoint from $L$.

Let $a + L \subseteq (A + L)/(L) \cap (G(L))$, $a = ng + y$, $g \subseteq G$, $y \subseteq L$. Now, since $A/K$ is pure-high in $G/K$, $a = M = ng + h$ implies $a = ng + k$, with $k \subseteq M \cap A = K$, and $n - y = ng + a - y$ so that $k + L = n(g + a) + L = n(k + L) \subseteq (K + L)/(L)$ and $k = nk$. Thus $a = n(k + L) \subseteq nA$ and $a + L \subseteq (A + L)/(L)$. Therefore $A$ is pure-high (with respect to $L$) in $G$.

Proofs of (d), (f), and (g) are similar and are omitted.

Properties (f), (g), (h), (i) are particularly important since, together with the results in [1], they yield most of

**Theorem 6.** Let $0 \to A \xrightarrow{L} B \xrightarrow{C} 0$ be pure-high exact. Then for any group $G$,

$$0 \to \text{Hom}(G, A) \to \text{Hom}(G, B) \to \text{Hom}(G, C) \to 0$$

and

$$0 \to \text{Hom}(G, A) \to \text{Hom}(G, B) \to \text{Hom}(G, C) \to 0$$

are exact.

Proof. Exactness of the sequences above, with the zeros at the end deleted, is established in [1], using (f), (g), (h), and (i) of Theorem 5.

Let $0 \to A \xrightarrow{L} E_1 \xrightarrow{D_1} 0$ and $0 \to C \xrightarrow{E_2} E_2 \xrightarrow{D_2} 0$ be pure injective resolutions of $A$ and $C$, respectively, with $D_1$ divisible. Since $A$ is pure in $B$ there exists a map $B \to E_1$ with $h = f$. Let $k : B \to E_1 \to E_2$ be defined by $k(0) = (h, b)$, $g(y)$. The map $k$ is clearly a monomorphism. Let $F = (E_2 \to E_2)/(k(B))$. It is easy checked that there exist maps $D_1 \to F \to D_1$ such that the diagram

$$\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}$$

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is commutative with exact rows and columns. One may also check that $k(B)$ is a pure subgroup of $E_i \oplus E_i$, so that diagram
\[
\begin{array}{ccc}
\text{Shom}(G, F) \to \text{Shom}(G, D_2) & \to & 0 \\
\downarrow & & \downarrow \\
\text{Hext}_1(B, G) \to \text{Hext}_1(G, C) & & 0
\end{array}
\]
is commutative with exact columns. Since $D_2$ is divisible the exact sequence $0 \to D_2 \to F \to D_2 \to 0$ is split exact, so that the top row of the diagram above is exact. Now a simple diagram chase yields the exactness of $\text{Hext}_1(B, G) \to \text{Hext}_1(G, C) \to 0$.

To prove exactness of $\text{Hext}_1(B, G) \to \text{Hext}_1(A, G) \to 0$, let $0 \to G \to E \to E/G \to 0$ be a pure injective resolution of $G$, with $E/G$ divisible. This yields a commutative diagram
\[
\begin{array}{ccc}
\text{Shom}(B, E(G)) \to \text{Shom}(A, E(G)) & \to & 0 \\
\downarrow & & \downarrow \\
\text{Hext}_1(B, G) \to \text{Hext}_1(A, G) & & 0
\end{array}
\]
with exact columns. We will prove the top row is exact, then a simple diagram chase yields the desired result. Let $f \in \text{Shom}(A, E(G))$. There is a subgroup $K$ of $B$ with $A$ max disjoint from $K$ in $B$. The natural map $A \to B/K$ is a monomorphism, and the induced exact sequence $\text{Hom}(B/k, E(G)) \to \text{Hom}(A, E(G)) \to \text{Ext}(B/(A \oplus K), E(G) = 0$ yields a map $g : B/K \to E/G$ with $g \alpha = f$. Let $\beta : B \to B/K$ be the natural map and $k = g \beta$. Then $h = g \beta = g \alpha = f$. Since $A \oplus K$ is essential in $B$ and $\ker f \oplus K$ is essential in $B$. Then the observation that $\ker h \subseteq \ker f \oplus K$ shows that $h \in \text{Shom}(B, E(G))$, proving that $\text{Shom}(B, E(G)) \to \text{Shom}(A, E(G)) \to 0$ is exact.

Theorem 2 generalizes to

**Theorem 7.** If $A$ is torsion, then $\text{Hext}_1(A, H) = \cap \{ \ker \text{Ext}(A, H) \}$

**Proof.** Let $0 \to H \to D$ be a divisible resolution of $H$, and $H \to \text{Ext}(Q(Z, H))/\ker \text{Ext}(Q(Z, H))$ the composition of the maps $Q(Z, H) \to \text{Ext}(Q(Z, H)) \to \text{Ext}(Q(Z, H))/\ker \text{Ext}(Q(Z, H))$. Then $0 \to H \to \ker \text{Ext}(Q(Z, H))/\ker \text{Ext}(Q(Z, H)) \oplus D = C$, where $\alpha(h) = (g(h), f(h))$, is a pure injective resolution of $H$. It is easily checked that $C/H$ is divisible. Now from Lemma 3, $\text{Shom}(A, C/H) = \cap \{ \ker \text{Hom}(A, C/H) \}$.

This fact, together with the exactness of $\text{Shom}(A, C/H) \to \text{Hext}_1(A, H) \to 0$, yields the desired result.

Finally, we give a direct proof of

**Theorem 8.** $\text{Hext}_1(X, Y)$ is an additive bi-functor.

**Proof.** Let $f : A \to B$ be a homomorphism. Given a group $G$, $f$ induces a homomorphism $\text{Ext}(G, A) \to \text{Ext}(G, B)$. Let $g$ be the restriction of $f$ to $\text{Hext}_1(G, A)$.
Let $0 \rightarrow A \rightarrow X \rightarrow G \rightarrow 0$ in $\text{Ext}_4(G, A)$. This sequence maps onto the extension $0 \rightarrow B \otimes X \rightarrow G \rightarrow 0$ where $Y = (B \otimes X)/M$ with $M = \{(f_0, -a) \mid a \in A\}$. There exists a subgroup $H$ of $X$ such that $A$ is maximal disjoint from $H$ and $(A + H)/M$ is pure in $X/H$. Let $K = \{(0, h) + M \mid h \in H\}$. The image of $Y$ in $Y$ is $B_1 = (B \otimes A)/M$. Clearly $B_1 \cap K = 0$. Suppose $B_1 \subset C \subset Y$ and $C \neq B_1$. Let $L = \{x \in Y \mid (0, x) + M \subset C\}$ for some $b \in B$. Then $A \subset L$, $A \neq L$ so $L \cap H = 0$. Let $h \in L \cap H$ with $h \neq 0$. For some $b \in B$, $h + M \subset C$ and, since $(0, h) + M \subset C$, $h + M \subset C$ and hence $0 = (0, h) + M + C \subset C \cap K$. Thus $B_1$ is maximal disjoint from $K$.

Let $(b, 0) + M = \{n((b, 1) + M) = \{(0, h) + M\}$ with $b \in B, x \in X, h \in H$ and, for some $a \in A$, $(b, n) = n((b, 1) + (0, h)) \subset \{f(g), -a\}$. Now since $A$ is pure-high in $X$ with respect to $H, a = nh$ implies $a \in nA$ so that $b = nb, f(a) \subset nA$. Therefore $(b, 0) + M = + A \cap n((b, 1) + M)$, and $B_1$ is pure-high with respect to $K$ in $Y$. This proves that $h: \text{Ext}_4(G, A) \rightarrow \text{Ext}_4(G, B)$.

The homomorphism $f: A \rightarrow B$ also induces a map $\text{Ext}(B, G) \rightarrow \text{Ext}(A, G)$. Let $h$ be the restriction of this map to $\text{Ext}_4(B, G)$. Let $0 \rightarrow G \rightarrow X \rightarrow B \rightarrow 0$ in $\text{Ext}_4(B, G)$. This sequence maps onto the extension $0 \rightarrow G \rightarrow Y \rightarrow A \rightarrow 0$ where $Y = [(x, a) \mid x \in X, a \in A, f(a) = n(x)]$. Let $H$ be a subgroup of $Y$ such that $G$ is pure-high with respect to $H$ and let $K = \{(x, 0) + (x, a) \mid x \in X, h \in H\}$. It is easily shown that $G$ is pure-high with respect to $K$ in $Y$. Thus $h: \text{Ext}_4(B, G) \rightarrow \text{Ext}_4(A, G)$.

It follows from this and properties inherited from $\text{Ext}(X, Y)$ that $\text{Ext}_4(X, Y)$ is an additive bi-functor.

4. Neat-high extensions

This section is devoted to extending further the concept of high extensions. It seems that the results of this section are more readily extensible to modules, although there are some difficulties involved. Most of the proofs here will be merely indicated, since they parallel the proofs of the previous section.

Definition 3. An extension $0 \rightarrow A \rightarrow G \rightarrow 0$ is a neat extension if and only if $A \subset K$ is an essential subgroup of $G$, implies $K/A$ is essential in $G/A$. Also, $A$ is called a neat subgroup of $G$.

This definition is equivalent to the usual definition of neatness ([3], p. 91). In fact, if $A$ is a subgroup of $G$, the following statements can readily be shown to be equivalent:

(a) $A$ is a neat subgroup of $G$.

(b) $A$ is maximal disjoint from some subgroup $K$ of $G$.

(c) If $K$ is a subgroup of $G$, maximal disjoint from $A$, then $A$ is maximal disjoint from $K$.

(d) $A \cap G = pA$ for all primes $p$.

A group $E$ is called neat injective if every neat exact sequence $0 \rightarrow E \rightarrow G \rightarrow H \rightarrow 0$ splits. (or equivalently $0 \rightarrow \text{Hom}(E, G) \rightarrow \text{Hom}(E, H) \rightarrow 0$ is exact) whenever $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is neat exact.

The following two lemmas are fairly well known, but no proofs seem to be in the literature.
LEMMA 4. A group $E$ is neat injective if and only if $E = D \bigoplus \bigoplus_{p} T_{p}$, where $D$ is divisible, $pT_{p} = 0$, and $p$ ranges over all primes.

PROOF. Suppose $E$ is neat injective. Let $E = D \oplus R$, where $D$ is divisible and $R$ is reduced. Let $T_{p}$ be the $p$-component of $R$. If the reduced $p$-group $T_{p}$ has an element of order $p^{2}$, then it has a cyclic summand $C(p^{2})$ of order $p^{n}$, $n \geq 1$ (e.g. a cyclic summand of a basic subgroup of $T_{p}$). Let $a$ and $b$ be cyclic groups generated by $a$ and $b$ with orders $p^{n+1}$ and $p^{n}$, respectively. Then the subgroup of $A \oplus B$ generated by $pa + pb$ is isomorphic to $C(p^{n})$ and is neat but not pure in $A \oplus B$. Now $R = C(p^{n}) \oplus S$ since $C(p^{n})$ is pure and of bounded order in $R$. We have the obvious neat exact sequence $0 \rightarrow E \rightarrow D \oplus A \oplus B \oplus S$ which does not split. Hence every element in $T_{p}$ has order $p$.

Suppose $R = X \oplus Y$, $X$ torsion free. Let $0 \rightarrow X \rightarrow D_{1}$ be exact with $D_{1}$ divisible. Then the sequence $0 \rightarrow X \rightarrow D_{1} \bigoplus \bigoplus_{p} (X/pX)$, where $g(x) = (f(x), (x+pX))$, is neat exact and must split. However, $X$ is torsion free reduced, and $D_{1} \bigoplus \bigoplus_{p} (X/pX)$ has no non-zero torsion free reduced summands. It follows that $X = 0$.

Clearly $R$ is cotorsion if any extension by a torsion-free group is neat. By what we have just shown, $R$ is adjusted so that $R = \text{Ext} (Q/Z, R)$. But $R = \bigoplus_{p} T_{p}$ with $pT_{p} = 0$, and $\text{Ext} (Q/Z, \bigoplus_{p} T_{p}) \cong \bigoplus_{p} \text{Ext} (Z/pZ, T_{p}) \cong \bigoplus_{p} T_{p}$. Thus $E$ has the desired form.

Now suppose $E = D \bigoplus \bigoplus_{p} T_{p}, pT_{p} = 0$, each $T_{p}$ is neat injective, since whenever $T_{p}$ is a neat subgroup it is a pure subgroup. Being of bounded order, it is then a summand of every group in which it is neat. Products of neat injectives are easily seen to be neat injective. It follows that $E$ is neat injective.

LEMMA 5. There are enough neat injectives. That is, every group is a neat subgroup of a neat injective.

PROOF. For a group $X$, let $0 \rightarrow X \rightarrow D \bigoplus \bigoplus_{p} (X/pX)$ with $g(x) = (f(x), (x + pX))$ be neat exact, and $D \bigoplus \bigoplus_{p} (X/pX)$ is neat injective by the previous lemma.

Lemma 2 holds for neat injective resolutions, the proof being completely analogous. If $0 \rightarrow A \rightarrow D \rightarrow 0$ is a neat-high extension, if and only if it is in the image of the map $\text{Shom} (B, E/A) \rightarrow \text{Ext} (B, A)$. This image is denoted $\text{Hex}_{\text{neat}} (B, A)$.

The proofs of the following theorems are completely analogous to the proofs of the corresponding theorems in Section 3, and hence are omitted.

THEOREM 9. The exact sequence $0 \rightarrow A \rightarrow G \rightarrow 0$ is a neat-high extension if and only if there exists a subgroup $K$ of $G$ such that $A$ is maximal disjoint from $K$ and $(A + K)/K$ is neat in $G/K$.

THEOREM 10. Let $A$ and $K$ be subgroups of $G$. The following hold.

(a) If $A$ is maximal disjoint from $K$ then $(A + K)/K$ is neat in $G/K$ (and hence $A$ is neat-high with respect to $K$), if and only if $K \cap (A - pG) = K \cap pA$ for all primes $p$. 


(b) If \( K \subseteq pG \) for all primes \( p \) and \( A \) is maximal disjoint from \( K \) then \( A \) is neat-high with respect to \( K \).
(c) If \( A \) is maximal disjoint from \( K \) and \( G/A \) is divisible, then \( A \) is neat-high with respect to \( K \) if and only if \( K \subseteq pG \) for all primes \( p \).
(d) If \( A \) is neat-high with respect to \( K \) then, in \( G \), \( A \) is neat-high with respect to \( K \).
(e) If \( A \) is neat-high with respect to \( K \) in \( G \), then \( \Sigma A \) is neat-high with respect to \( \Sigma K \) in \( \Sigma G \).
(f) If \( A \) is neat-high in \( G \) and \( A \triangleleft K \subseteq G \), then \( A \) is neat-high in \( K \).
(g) If \( A \) is neat-high in \( G \) and \( K \triangleleft A \), then \( A/K \) is neat-high in \( G/K \).
(h) If \( A \) is neat-high in \( K \) and \( K \) is neat-high in \( G \), then \( A \) is neat-high in \( G \).
(i) If \( K \) is neat-high in \( G \), and \( A/K \) is neat-high in \( G/K \), then \( A \) is neat-high in \( G \).

**Theorem 11.** Let \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) be neat-high exact. Then for any group \( G \),
\[
0 \rightarrow \text{Hom}(G, A) \rightarrow \text{Hom}(G, B) \rightarrow \text{Hom}(G, C) \rightarrow 0
\]
and
\[
0 \rightarrow \text{Hext}_1(G, A) \rightarrow \text{Hext}_1(G, B) \rightarrow \text{Hext}_1(G, C) \rightarrow 0
\]
are exact.

**Theorem 12.** \( \text{Hext}_1(X, Y) \) is an additive bi-functor.

### 5. Some properties of Shom

The subgroup \( \text{Shom}(A, B) \) of \( \text{Hom}(A, B) \) has played a fundamental role in our discussion of high extensions. In this final section, we note some additional functorial properties of \( \text{Shom} \).

By routine arguments, it can be shown that \( \text{Shom}(A, B) \) is an additive functor, and considered as a functor of \( B \) is left exact. It is not left exact as a functor of \( A \). For this reason, from now on \( \text{Shom}(A, B) \) will be considered as a functor of the single variable \( B \). Let \( \text{Next}(A, B) \) be the right derived functor of \( \text{Shom}(A, B) \), and let \( \text{Ext}(A, B) \) be the set of neat extensions in \( \text{Ext}(A, B) \).

**Theorem 13.** Let \( 0 \rightarrow B \rightarrow D \rightarrow B \rightarrow 0 \) be an injective resolution of \( B \). Then there exist a natural homomorphism \( h \) such that the diagram
\[
\begin{array}{ccc}
\text{Shom}(A, D/B) & \xrightarrow{f} & \text{Next}(A, B) \\
\uparrow & & \uparrow \\
\text{Shom}(A, D/B) & \xrightarrow{g} & \text{Ext}(A, B)
\end{array}
\]
is commutative with exact rows, where \( f \) is the restriction of the map from \( \text{Hom}(A, D/B) \) into \( \text{Ext}(A, B) \), and \( g \) is the usual connecting homomorphism.

**Proof.** First we show that the top row is exact. Let \( \alpha \in \text{Shom}(A, D/B) \). Then \( f(\alpha) \) is the sequence \( 0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0 \) where \( X = \{(d, a) \mid d + B = \alpha(a)\} \), \( \beta(B) = \{(b, o)\} \), and \( \gamma(d, a) = a \). (We are considering, of course, that \( B \) is actually a subgroup of \( D \)) It is easy to show that \( \beta(B) \) is maximal disjoint from the subgroup
\[ (0, \alpha) \in \ker \delta \) of \( X \), and hence that the sequence is a neat extension. Thus \( \ker f \) is independent of the particular injective resolution of \( B \) takes. Now suppose \( 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0 \) is neat exact. Then \( B \) is maximal disjoint from a subgroup \( K \) of \( X \). Let \( 0 \rightarrow X \rightarrow E \rightarrow X \rightarrow 0 \) be an injective resolution of \( X \). Then we have injective resolutions \( 0 \rightarrow \epsilon \rightarrow \epsilon \rightarrow \epsilon \rightarrow 0 \) and \( 0 \rightarrow (B + K) \rightarrow \epsilon \rightarrow \epsilon \rightarrow 0 \). Also, \( A \cong X/B \), and we have maps \( A \rightarrow \epsilon \rightarrow \epsilon \rightarrow \epsilon \rightarrow 0 \), the composition of which gives \( \sigma \in \operatorname{Ext}(A, E) \). This map \( \sigma \) is in \( \operatorname{Shom}(A, E) \) since \( B \) is neat in \( X \), and \( (B + K) \) is essential in \( X/B \), hence \( \ker \sigma \) is essential in \( X/B \). By an argument entirely analogous to the one used in the proof of Theorem 1, it follows that the image of \( \sigma \) in \( \operatorname{Ext}(A, E) \) is our neat sequence \( 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0 \). Thus \( \operatorname{Shom}(A, B) \rightarrow \operatorname{Next}(A, B) \rightarrow 0 \) is exact.

The observation that \( \ker g \subset \ker f \) yields a map \( h : \operatorname{Shom}(A, B) \rightarrow \operatorname{Next}(A, B) \), as desired.

An immediate corollary is, of course, that \( \operatorname{Next}(A, B) \) is a subgroup of \( \operatorname{Ext}(A, B) \), and that \( X \) is an epimorphism.

Let \( \operatorname{Sex}^i(A, B) \) be the right derived functor of \( \operatorname{Shom}(A, B) \). That is, if \( 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0 \) is a neat injective resolution of \( E \), then \( \operatorname{Sex}^i(A, B) \cong \operatorname{Ext}^i(A, E) \) in the sequence \( \operatorname{Shom}(A, E) \). Analogous to Theorem 1, we have the diagram

\[ \operatorname{Shom}(A, E/B) \xrightarrow{h} \operatorname{Next}^i(A, B) \rightarrow 0 \]

\[ \operatorname{Shom}(A, E/B) \xrightarrow{i} \operatorname{Sex}^i(A, B) \rightarrow 0 \]

which is commutative and has exact rows. From the definitions of \( \operatorname{Next}^i \) and \( \operatorname{Sex}^i \), the rows are exact. Again \( h \) is obtained by observing that \( \ker g \subset f \). The analogous remarks hold for \( \operatorname{Sex}^i(A, B) \), the pure right derived functor of \( \operatorname{Shom}(A, B) \).

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