Some Basic Theory of Interval-Valued Fuzzy Sets

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Abstract
Let $S$ be a set. A fuzzy subset of $S$ is a mapping $A$ from $S$ into $[0,1]$. The value $A(a)$ for a particular $a$ is typically associated with a degree of belief of some expert. There is an extensive theory of fuzzy sets. But there are situations when assigning an exact number to an expert opinion is too restrictive, and the assignment of an interval of values is more realistic. This paper is concerned with the basics of a theory for such "interval-valued" fuzzy sets, namely, mappings of a set $S$ into the set of intervals of $[0,1]$.

1. Interval-Valued Fuzzy Sets

In interval-valued fuzzy set theory, the interval $[0,1]$ is replaced by the set of subintervals of $[0,1]$. Since an interval is completely determined by its endpoints, this set of intervals can be identified with the set $(a,b) : a, b \in [0,1]$, $a \leq b$. The element $(a,b)$ is just the pair with $a \leq b$. We will refer to these pairs as intervals. In lattice theory, the standard notation for this set is $[0,1]$. So an interval-valued fuzzy subset of a set $S$ is a mapping $A : S \rightarrow [0,1]$.

Now comes the crucial question: What structure should $[0,1]$ be endowed with? Again, lattice theory provides an answer. We use componentwise operations coming from the operations on $[0,1]$. For example, $(a,b) \leq (c,d)$ if $a \leq c$ and $b \leq d$, which gives the usual lattice meet and join operations

$$(a,b) \lor (c,d) = (a \lor c, b \lor d)$$

$$(a,b) \land (c,d) = (a \land c, b \land d)$$

The point $(0,0)$ is the smallest element of this lattice and $(1,1)$ is the largest element. The resulting lattice has the standard notation $I^{I^*}$, where $I = (0,1]$. This is an algebra $(I^*, \land, \lor, 0, 1)$ with componentwise operations $\land$ and $\lor$ and the constants $0 = (0,0)$ and $1 = (1,1)$. This is a fundamental lattice-theoretical construction: a lattice $L$, form $I^I$ and use componentwise operations. The resulting lattice has many of the same properties as the original lattice.

In particular, if $L$ is a complete distributive lattice, then so is $I^L$. The proposal here is to use the algebra $I^I$ as the basic building block for interval-valued fuzzy set theory.

The negation $\alpha' = 1 - \alpha$ on $[0,1]$ induces the negation on $I^I$ given by $(a,b)' = (b,a')$. With this negation, $I^I$ becomes a De Morgan algebra $(I^I, \lor, \land)$, that is,

* $(I^I, \lor, \land)$ is a bounded, distributive lattice,

* with a negation $'$ satisfying the De Morgan laws with respect to the lattice operations $\land$ and $\lor$.

This in turn yields the structure of a De Morgan algebra for the set of all interval-valued fuzzy subsets of $S$ with the operations

$$(A \land B)(x) = A(x) \land B(x)$$

$$(A \lor B)(x) = A(x) \lor B(x)$$

$$(A')(x) = (A(x))'$$

2. Automorphisms of the Lattice of Subintervals of the Unit Interval

A good share of the theory of fuzzy sets is concerned with endowing $I$ with additional structure such as a linear order, a lattice, and a negation other than the usual ones. Indeed, the approach given in this paper is to consider $I^I$ as a lattice with additional structure and study the properties of its sublattices. In particular, the automorphism group of $I^I$ is a fundamental object in the theory of fuzzy sets.

Definition 1. An automorphism of $I^I$ is a one-to-one map $f$ from $[0,1]$ onto itself such that $f(0) \leq f(x)$ if and only if $x \leq y$. An anti-automorphism is a one-to-one map $f$ from $[0,1]$ onto itself such that $f(x) \leq f(y)$ if and only if $x \geq y$.

The set of all automorphisms of $I^I$ is denoted $Aut(I^I)$ and the set of all automorphisms and anti-automorphisms is denoted $Map(I^I)$. Both of these are
groups under composition of maps, and $\text{Aut}(\mathbb{F}_2)$ is a normal subgroup of index 2 in $\text{Map}(\mathbb{F}_2)$. In the plane, $\{0,1\}$ is the triangle pictured. Each leg is mapped onto itself by automorphisms.

Lemma 2 Let $A = \{(0,a) : a \in \{0,1\}\}$, $B = \{(b,1) : b \in \{0,1\}\}$, and $C = \{(c,c) : c \in \{0,1\}\}$. If $f \in \text{Aut}(\mathbb{F}_2)$ then $f(A) = A$, $f(B) = B$, and $f(C) = C$.

Theorem 3 [2] Every automorphism $f$ of $\mathbb{F}_2$ is of the form $f(a,b) = (g(a),g(b))$, where $g$ is an automorphism of $\mathbb{F}_2$. Thus, the groups $\text{Aut}(\mathbb{F}_2)$ and $\text{Aut}(\mathbb{F}_2)$ are isomorphic.

Let $\alpha$ be the anti-automorphism of $\mathbb{F}_2$ given by $\alpha(a) = 1 - a$. Then, $(a,b) \mapsto (\alpha(a),\alpha(b))$ is an anti-automorphism of $\mathbb{F}_2$ which we also denote by $\alpha$. If $g$ is an anti-automorphism of $\mathbb{F}_2$, then $g = \alpha f$ for some automorphism $f = \alpha g$. Thus we have a similar theorem for maps. Theorem 4 Every anti-automorphism of $\mathbb{F}_2$ is of the form $(a,b) \mapsto (g(a),g(b))$, where $g$ is an anti-automorphism of $\mathbb{F}_2$. The groups $\text{Map}(\mathbb{F}_2)$ and $\text{Map}(\mathbb{F}_2)$ are isomorphic.

These results are in [2]. Further facts about automorphisms and anti-automorphisms follow from results in [1]. In particular, we have:

Theorem 5 The involutions of $\mathbb{F}_2$ are precisely the involutions $f^{-1}af : f \in \text{Aut}(\mathbb{F}_2)$.

The following relations on $\{0,1\}^{[2]}$ are considered by Zue, Nguyen, Kleinov, et al., in [4, 7, 6].

- $\alpha(a,b) \leq (a \circ b) \in \{0,1\}^{[2]}$ if $a \leq b \leq 1/2$.
- $\alpha(a,b) \leq (c,d)$ if $c \leq a \leq b \leq d$.

They create $((a,d) \leq (b,c))$ possibly less than $((a,b) \in \{0,1\})$ for $((a,d) \leq (b,c))$ does hold. The other relations like $\leq$ yield the same set of automorphisms of $\mathbb{F}_2$ as the lattice order $\leq$.

Theorem 6 A bijection $f : [2]^2 \to [2]^2$ preserves the natural lattice order if and only if it preserves the natural lattice order.

Proof. Assume that $f(x) \leq f(y)$ whenever $x \leq y$, and $f(x) \leq f(y)$ whenever $x \leq y$. Suppose $x \leq y$. If $x = y$, then $f(y) = f(y)$ implies $f(x) \leq f(y)$, if $x$ is not $y$, let $x = (a,b)$ and $y = (c,d)$. We have $x \leq y$, where $a \leq (a,b)$ and $(c,d) \leq (c,d)$. Also, $(a,b) \leq (c,d)$ and $(c,d) \leq (c,d)$. Thus $f(a,b) \leq f(c,d)$. Then $f(a,b) \leq f(c,d)$ and $f(c,d) \leq f(c,d)$. The first line implies that the left endpoint of $f(a,b)$ is less than or equal to the left endpoint of $f(c,d)$, and the second line implies that the right endpoint of $f(x,b)$ is less than or equal to the right endpoint of $f(c,d)$. Thus the bijection $f$ preserves the lattice order, implying that it is an automorphism of $\mathbb{F}_2$.

The converse follows easily from Theorem 3.

3. T-norms for Interval-Valued Fuzzy Sets

The first problem with T-norms is just what their definition would be. A T-norm on $[0,1]$ should be a generalization of T-norm on $[0,1]$ and should satisfy the appropriate monotonicity and boundary conditions. In [2] we made the following definition and states and proved Theorem 8. Much of the theory for T-norms carries over to the interval-valued case.

Definition 7 A commutative, associative binary operation $\otimes$ on $[0,1]$ is a t-norm if for all $x, y \in [0,1]$ and $x, y \in [0,1]$ we have:

1. $x \otimes y = y \otimes x$ where, where $x = (x,y) \in [0,1]$.
2. $x \otimes (y \otimes z) = (x \otimes y) \otimes (x \otimes z)$
3. $x \otimes (y \otimes z) = (x \otimes y) \otimes (x \otimes z)$
4. $(1,1) = 1 = x$
5. $(0,1) = (0,0)$

Theorem 8 Every t-norm on $[0,1]$ is of the form

$(x,y) \otimes (x,y) = (x \circ u, y \circ u)$

where $(0,1) \otimes (0,1) = (0,1)$

Several additional useful properties follow immedi-
ately from the t-norm $\otimes$ on $[0,1]$. 

\[ 0.785, 0.435, 1.0, 0.895 \]
1. \( x \) is increasing in each variable.
2. \( x \cap y \leq x \land y \)
3. \((0,0) \leq (0,0)\)
4. \((0,0) \leq (0,0)\) for some \( c \).
5. The restriction of \( \otimes \) to \( A, B, \) or \( C \) induces a t-norm on \( \mathcal{I} \).

In [6], Q. Zhao stated the following definition. Here, we use the notation \((a, b) \leq (c, d)\) if \( b \leq c \) and \((a, b) \subset (c, d) \) if \( b < c \).

**Definitions**

1. \( x \otimes y = 0 \Rightarrow x = 0 \) and \( x \otimes 1 = 1 \Rightarrow x = x \)
2. If \( a < c \) and \( b < d \) then \( a \otimes b < x \).
3. If \( a < c \) and \( y \otimes z \leq x \) then \( a \otimes b \leq x \).
4. If \( a \leq x \) and \( b \leq y \) then \( a \otimes b \leq x \).

A t-norm is a generalized AND operation; so each it should be a commutative, associative, binary operation with a unit, preserving monotonicity. The property of monotonicity is clearly very important for a t-norm. In the fuzzy-set case, the notion is incomplete—the there is only one order aspect, and a t-norm is defined to be a binary operation that is increasing in both variables (with respect to the standard order on the unit interval). In the interval-valued case, the situation is more complicated, and monotonicity is the property that has been offered in different forms. In [2] we require monotonicity in terms of distributive laws based on the lattice-order.

1. \((a, b) \leq (c, d)\) if \( a \leq c \) and \( b \leq (c, d)\)

Xiao [6] and Nguyen and Klementich [4,1] require monotonicity for the following relations.

1. \((a, b) \leq (c, d)\) if \( a \leq b \leq c \leq d \) (inclusion)
2. \((a, b) \leq (c, d)\) if \( a \leq b \leq c \leq d \) (strict inclusion)
3. \(a \otimes b \leq x \leq d \leq c \leq d \) (strictly necessary)

Also, they consider only the cases of continuous t-norms. In all these cases, a t-norm defined in terms of monotonicity with respect to the chosen relation(s) has been shown to be one that comes directly from a t-norm on fuzzy sets applied compositionwise, that is, a t-norm on interval-valued fuzzy sets was shown to be of the form

\[
(a, b) \otimes (c, d) = (a \circ c, b \circ d)
\]

for some appropriate t-norm

\[
\circ : [0,1] \times [0,1] \rightarrow [0,1]
\]

The following theorem shows that the combination of lattice order and inclusion has some interesting implications.

**Theorem 10**

Let \( \otimes \) be a commutative, associative, binary operation on \( \mathcal{I} \) that preserves lattice order and inclusion. Then \( \otimes \) distributes over and over, that is,

\[
x \otimes (y \land z) = (x \otimes y) \land (x \otimes z)
\]

for all \( x, y, z \in \mathcal{X} \). If \( \otimes \) also satisfies

\[
(1,1) \otimes x = x
\]

for all \( x \in \mathcal{X} \), then it satisfies

\[
(0,0) \leq x = (0,0)
\]

\[
(0,1) \otimes (0, b) = (0, b)
\]

for all \( a, b \in \mathcal{I} \).

**Proof.** Since \( a \land x \leq y \) and \( y \land x \leq x \), we have

\[
x \otimes (y \land z) \leq (x \otimes y) \land (x \otimes z)
\]

Let \( y = (a, b) \) and \( x = (c, d) \). Since both \( \otimes \) and \( \land \) are commutative, we may assume without loss of generality that \( a \leq c \). Then either \( b \leq d \) or \( c \geq d \). In the first case, \( y \land x = (a, b) \), whence \( x \otimes y \leq x \otimes z \) so we have

\[
x \otimes (y \land z) = x \otimes y \leq (x \otimes y) \land (x \otimes z)
\]

In the second case, \( a \leq c \leq b \leq d \leq c \leq d \) whence \( x \otimes x \leq x \). Thus

\[
x \otimes x \leq x \otimes (y \land z) \leq x \otimes y
\]

From one of the observations above, this latter containment together with

\[
x \otimes (y \land z) \leq (x \otimes y) \land (x \otimes z)
\]

implies that the two intervals \( x \otimes (y \land z) \) and \( (x \otimes y) \land (x \otimes z) \) have the same left and right endpoints, so they are equal.

The proof of the other distributive law is similar.

Assume that \((1,1) \otimes x = x \) for all \( x \in \mathcal{I} \). Then \( x \leq (1,1) \) implies

\[
(0,0) \otimes x \leq (0,0) \otimes (1,1) = (0,0)
\]
\[(0, 0) \times x = (0, 0) \text{ for all } x \in \mathbb{R}. \] Also
\[(1, 1) \subseteq (0, 1) \subseteq (1, 1)\]
implies
\[x = (1, 1) \otimes x \subseteq (0, 1) \otimes x \subseteq (1, 1) \otimes x = x \]
Let \(a = (a, b)\) and \((0, 1) \otimes x = (c, d)\). The containment implies that \(c \leq a, c \leq b \leq d\), and the inequality implies that \(a \leq b, b = d\). Thus \(x = a\). Now
\[(0, 0) = (0, 0) \otimes x \subseteq (0, 1) \otimes x \]
forges the left endpoint of \((0, 0) \otimes x \) to be \(0\), consequently
\[(0, 1) \otimes (a, b) = (0, 0)\]
This gives the following theorem.

**Theorem 11** A commutative, associative, binary operation \(\otimes : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}^2\) satisfying
1. \(x \otimes (1, 1) = x\) for all \(x \in \mathbb{F}^2\),
2. \(\otimes\) preserves the lattice order,
3. \(\otimes\) preserves inclusion, and
4. \(C \subseteq C \subseteq C^\prime\) where \(C \times [c, c) : 0 \leq c \leq 1\) is the diagonal,
is a t-norm on \(\mathbb{F}^2\), that is, satisfies Definition 7.

**Proof.** By Theorem 10, an operation satisfying these conditions also satisfies the conditions of Definition 7. The converse can be easily verified from the results in Theorem 11 above.

Most of the t-norms we will consider are counts in the sense of the following definition. On \(1, 0, \mathbb{F}^2\), they are the exact count of t-norms.

**Definition 12** A binary operation \(\otimes : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}^2\) is convex if, given \(a, b \leq x \leq y \leq c\), there exists \(x = u \otimes v\), where \(u \leq x \leq v\), such that \(x \leq u \otimes v\).

We define Archimedean, unit, and nilpotent t-norms on \(\mathbb{F}^2\) just as for t-norms on \(1, 0\). In the context of a t-norm \(\otimes\),
we will write \(x = a \otimes \cdots \otimes a\).

**Definition 13** A t-norm \(\otimes : \mathbb{F}^2 \to \mathbb{F}^2\) is Archimedean if, for given \(x, y \in \mathbb{F}^2\) with \(x \leq y\), there is a positive integer \(n\) with \(x^n \leq y\). A convex Archimedean t-norm is strict if \(x \otimes x = 0\) only for \(x = 0\) and nilpotent otherwise.

The characterization of convex (continuous) Archimedean t-norms on \(\mathbb{F}^2\) is analogous to that for \(1, 0\).

**Proposition 14** A convex t-norm \(\otimes : \mathbb{F}^2 \to \mathbb{F}^2\) is Archimedean if and only if it satisfies \(x \otimes x \in \mathbb{F}^2\) for all \(x \in \mathbb{F}^2, (0, 0), (0, 1), (1, 1)\).

A convex Archimedean t-norm on \(\mathbb{F}^2\) is nilpotent if and only if, for each \(a \in \mathbb{F}^2\), there is a positive integer \(n\) such that \(a^n \in \mathbb{F}^2\). The corresponding condition for a nilpotent convex Archimedean t-norm on \(\mathbb{F}^2\) is that for each \(a \in \mathbb{F}^2\), there is a positive integer \(n\) such that \(a^n \in \mathbb{F}^2\).

It is easy to see that, in the notation of Theorem 8, a convex t-norm \(\otimes\) is Archimedean, strict, or nilpotent if and only if the t-norm \(\otimes\) is Archimedean, strict, or nilpotent, respectively, in effect, the theory of t-norms on \(\mathbb{F}^2\) as we have defined them is reduced to the theory of t-norms on \(1, 0\).

### 4. Negations and T-conorms for Interval-Valued Fuzzy Sets

As an anti-autonomous \(J\) such that \(J(f(x)) = x\) is an inversion, or strong negation. The map given by \(\alpha \circ J = 1 - \beta\) is a negation, as is \(f^{-1} - J\) for any homomorphism \(f\), and there are no others. An\- autonemorphisms interchanges \((0, 0)\) and \((1, 1)\). Just as for \(1\), we define a t-concept to be the dual of a t-norm with respect to some negation.

**Definition 15** Let \(\otimes\) be a binary operation and \(\eta\) a negation on \(\mathbb{F}^2\). The dual of \(\otimes\) with respect to \(\eta\) is the binary operation \(\@\) given by
\[x \@ y = \eta(\eta(x) \otimes \eta(y))\]
If \(\otimes\) is a t-norm, then \(\@\) is called a t-conorm.

**Theorem 16** Every t-conorm \(\@\) on \(\mathbb{F}^2\) is of the form
\[(a, b) \@ (c, d) = (a + c, b + d)\]
where \(a \leq b\) and \(c \leq d\).

The proof follows by duality from the proof for t-norms. Thus the theory of t-conorms and negations on \(\mathbb{F}^2\) has also been reduced to that theory on \(1, 0\).

### 5. Conclusions

A definition is given for a t-norm on \(\{0, 1\}^2 = \{(a, b): a, b \in \{0, 1\}, a \leq b\}\) in terms of the usual lattice order on \(\{0, 1\}^2\), namely, \((a, b) \leq (c, d)\) if and only if \(a \leq c\) and \(b \leq d\). Such a t-norm is shown to be of the form \((a, b) \times (c, d) = (a + c, b + d)\) for some t-norm on \(\{0, 1\}\). In \(\{0, 1\}^2\), three binary relations are defined on \(\{0, 1\}^2\), and a continuous t-norm is defined requiring the preservation of these relations. In this connection, these two
definitions are equivalent. It is also shown that $[0, 1]^{10}$ with its natural composition order has the same monotonic group as $[0, 1]^{10}$ equipped with the binary relations in [6].

References


