MODULES OVER PIDs THAT ARE INJECTIVE
OVER THEIR ENDOMORPHISM RINGS

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1. Introduction

Let $E$ be any ring. An exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

divides $E$-modules is called pure if

$$0 \rightarrow A \otimes_E M \rightarrow 0 \rightarrow M \rightarrow 0$$

is exact for all left $R$-modules $M$. A right $E$-module $D$ is pure-injective if

$$0 \rightarrow \text{Hom}_E(C, D) \rightarrow \text{Hom}_E(B, D) \rightarrow \text{Hom}_E(A, D) \rightarrow 0$$

is exact for all pure exact sequences

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$
of right $E$-modules.

An abelian group is called algebraically compact if it is pure injective as a
$Z$-module. This is one of the few finitely closed classes of abelian groups. A complete
structure theory is presented in [1], for example. There the rather remarkable fact is
pointed out that an injective module over any ring $E$ is algebraically compact in an
abelian group. A more general surprise is our Corollary 1 to Lemma 5 below. The
additive group of any pure-injective $E$-module is algebraically compact. The question of deciding which (algebraically compact) groups can be additive groups of injective modules remains open, but in this paper we characterize those abelian groups $G$ that are injective when viewed as modules over their endomorphism rings $\text{End}(G)$.

We state in that group theory generalize untouchable to modules over a principal ideal domain, there are some curious differences in our present endeavor. Moreover, even were we restricted to groups, we would find ourselves looking at modules over the $p$-adic integers. For these reasons, we shall consider the problem over an arbitrary principal ideal domain $\mathcal{R}$.

Let $\mathcal{R}$ be a principal ideal domain and $\mathcal{K}$ its quotient field. If $p \in \mathcal{R}$ is a prime, the completion of $\mathcal{R}$ in the $p$-adic topology is denoted by $\mathcal{R}_p$, and its quotient field is denoted by $K_p$ and the rank-one divisible torsion $\mathcal{R}_p$-module $K_p/\mathcal{R}_p^\times$ is denoted by $\mathcal{K}_p$. When we indicate a product over the primes in $\mathcal{R}$, it is understood that we are selecting one prime from each associate class.

Theorem. Let $\mathcal{R}$ be a principal ideal domain and $\mathcal{G}$ an $\mathcal{R}$-module. Then $\mathcal{G}$ is injective as a module over its endomorphism ring if and only if $\mathcal{G} = K_p \mathcal{G}_p \oplus \mathcal{D}$, where $\mathcal{G}_p$ is a finite direct sum of cyclic $K_p$-modules, $\mathcal{D}$ is a finite-rank divisible $\mathcal{R}$-module, and either

1. $\mathcal{D} = 0$ and $\mathcal{G}_p$ is torsion for all $p$,
2. $\mathcal{D}$ is torsion-free for all $p$, and $\mathcal{G}_p = 0$ for all but finitely many $p$, or
3. $\mathcal{R}$ is a complete discrete valuation ring and the torsion submodule of $\mathcal{D}$ is injective.

The remainder of this paper is devoted to proving this theorem.

2. Duality

The basic building blocks are the rank-one $\mathcal{R}_p$-modules. Such a module is isomorphic to $\mathcal{R}_p \mathcal{K}_p, \mathcal{R}_p \mathcal{K}_p^\times$, or $\mathcal{R}_p^\times \mathcal{R}/\mathcal{R}^\times$. We shall construct injectives by constructing projectives and dualizing.

Lemma 1. Let $\mathcal{G}$ be a finite direct sum of rank-one $\mathcal{R}_p$-modules. Then $\mathcal{G}$ is injective as an $\mathcal{R}$-module if $\mathcal{G}$ is either reduced, or has a summand isomorphic to $\mathcal{R}_p$.

Proof. In either case we can find an element $x$ that generates a summand of $\mathcal{G}$ and can be mapped onto any element of $\mathcal{G}$ by an endomorphism. Let $x \in \mathcal{Y} = \text{End} \mathcal{G}$ be a projection on this summand and consider the map $\text{End} \mathcal{G} \rightarrow \mathcal{G}$ induced by evaluation at $x$. Since $x^* = x$, and $x$ can be mapped onto any element of $\mathcal{G}$, $\mathcal{G}$ is injective. On the other hand, if $\mathcal{G}(x) = \mathcal{G}$, then $x^* = x$ for all $x \in \mathcal{G}$, and $\mathcal{G}$ is one-to-one. Clearly $\mathcal{G}$ is an $\mathcal{R}$-endomorphism. Hence $\mathcal{F}^*$ is monomorphic to $\mathcal{G}$ as an $\mathcal{R}$-module. But, since $x = x$
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admitting, $E$ is a summand of $R$ and hence projective.

To go from projective to injective we look at $G = \text{Hom}_{\mathcal{S}}(G, \text{R}_{\text{proj}})$. Then $R^* = \text{R}_{\text{proj}} \otimes \text{R}_{\text{proj}} \cong \text{R}_{\text{proj}} \otimes \text{R}_{\text{proj}} \cong \text{R}_{\text{proj}} \otimes \text{R}_{\text{proj}}$ if $G$ is any finite sum of rank-one $\text{R}_{\text{proj}}$-modules. Thus the natural map $G \to G^*$ is an isomorphism. The significant property of $\text{R}_{\text{proj}}$ is that it is an injective cogenerator in the category of $\text{R}_{\text{proj}}$-modules. That is, $\text{R}_{\text{proj}}$ is injective and, if $M$ is a nonzero $\text{R}_{\text{proj}}$-module then $\text{Hom}_{\mathcal{S}}(M, \text{R}_{\text{proj}}) \neq 0$. This gives us a way of constructing injective modules.

Lemma 2. Let $E$ and $S$ be rings, and $T$ an injective cogenerator in the category of right $S$-modules. If $M$ is a left $E$-right $S$-bimodule, then $M^* = \text{Hom}_{S}(M, T)$ is an injective right $E$-module if and only if $M$ is flat as a left $E$-module.

Proof. $M$ is flat if and only if the sequence

$$0 \to A \otimes_E M \to A \otimes_E M \to C \otimes_E M \to 0$$

is exact for all exact sequences $0 \to A \to B \to C \to 0$ of right $E$-modules. But since $T$ is an injective cogenerator, (1) is exact if and only if the sequence

$$0 \to (C \otimes_E M)^* \to (B \otimes_E M)^* \to (A \otimes_E M)^* \to 0$$

is exact. But (2) is equivalent to

$$0 \to \text{Hom}_{S}(C, M^*) \to \text{Hom}_{S}(B, M^*) \to \text{Hom}_{S}(A, M^*) \to 0$$

by the natural equivalence $\text{Hom}_{S}(X, \text{R}_{\text{proj}} Y, Z) \cong \text{Hom}_{S}(X, \text{R}_{\text{proj}} Y, Z)$. But (3) holds for an arbitrary exact sequence $0 \to A \to B \to C \to 0$ of right $S$-modules if and only if $M^*$ is an injective right $E$-module.

Lemma 3. Let $G$ be a finite direct sum of rank-one $\text{R}_{\text{proj}}$-modules. Then $G$ is injective as an $E(G^*)$-module if and only if $G$ is torsion, or $G^*$ contains a copy of $\text{R}_{\text{proj}}$.

Proof. Let $G^* = \text{Hom}_{\mathcal{S}}(G, \text{R}_{\text{proj}})$. Then $G^*$ satisfies the hypotheses of Lemma 1, and is a projective if it is a left $E(G^*)$-module. By Lemma 2, $G^*$ is injective as a right $E(G^*)$-module. Now $G$ is naturally isomorphic, as a $\text{R}_{\text{proj}}$-module, to $G^{**}$; we must show that, under this isomorphism, the left $E(G^*)$-module structure of $G$ coincides with the right $E(G^*)$-module structure of $G^{**}$. Consider the following diagram:

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The maps $\alpha$, $\beta$, and $\gamma$ are the natural anti-homomorphisms induced by the contravariant functor $\mathbf{R}$. The column equalities are induced by the natural isomorphism $A \to A^{**}$. One must check that the diagram commutes, and observe that $\beta$ is the source of the right $E(G^{**})$-module structure on $G**$.

In Lemma 3, $E(G)$ is the endomorphism ring of $G$ as an $R_\alpha$-module. However, if $G$ is torsion, or if the rank of $R_\alpha$ is 1, then the endomorphism ring of $G$ as an $R_\alpha$-module coincides with the endomorphism ring of $G$ as an $R$-module. Hence Lemma 3 provides us with a class of $R$-modules that are injective as torsion-free modules over these endomorphism rings. A second class is provided by the finite-rank torsion-free divisible $R$-modules. If $G$ is such a module, then $E(G)$ is isomorphic to the endomorphism ring of $G$ as an $R$-module. Hence, as is well known, every $E(G)$-module is injective. In particular, $G$ is injective as a module over $E(G)$. These two basic classes may be combined to accordance with the following lemma.

**Lemma 4.** Let $G = \Pi G_\alpha$, and $\text{Hom}(G/G_\alpha, G_\alpha) = 0$ for all $\alpha$. Then $E(G) = \bigoplus E(G_\alpha)$, and $G_\alpha$ is injective over $E(G_\alpha)$ if and only if $G_\alpha$ is injective over $E(G_\alpha)$ for all $\alpha$.

**Proof.** First, $E(G) = \text{Hom}(G, G) = \Pi \text{Hom}(G_\alpha, G_\alpha) = \Pi (\text{Hom}(G_\alpha, G_\alpha) \otimes \text{Hom}(G_\alpha, G_\alpha)) = \Pi \text{Hom}(G_\alpha, G_\alpha) = \Pi E(G_\alpha)$. In particular, each $G_\alpha$ is an $E(G)$-module, so $G$ is injective as an $E(G)$-module if and only if $G_\alpha$ is injective as an $E(G)$-module for all $\alpha$, and it is readily verified that $G_\alpha$ is $E(G)$-injective if and only if it is $E(G_\alpha)$-injective.

We now have half of the theorem.

**Corollary.** Let $G_\alpha$ be a finite direct sum of cyclic $R_\alpha$-modules, and $G$ a finite-rank torsion-free $R$-module. Then the $R$-module $\bigoplus G_\alpha$ is injective over its endomorphism ring if either

1. $D = 0$ and $G_\alpha$ is torsion for all $\alpha$.
2. $D$ is torsion, $G_\alpha$ is torsion for all $\alpha$, and $G_\alpha = 0$ for all but finitely many $\alpha$, or
3. $R$ is a complete discrete valuation ring, and the torsion submodules of $D$ are nonzero.

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Proof. Such $R$–modules may be obtained by putting together a & Lemma 4, finite-rank torsion-free divisible $R$–modules, and the modules of Lemma 3.

Notice that we do not allow $D \neq 0$ if $C_0 \neq 0$ for infinitely many $p$. Indeed, if $C_0 \neq 0$ for infinitely many $p$, then there is a map from $\mathbb{F}_p$ onto $K$, proving any application of Lemma 4. Similarly, we do not mix torsion and torsion-free divisible $R$–modules unless $R$ is a complete discrete valuation ring. Showing that these restrictions are necessary is the subject of the remainder of this paper.

3. Pure-injectives

We call an $R$–module $G$ algebraically compact if $G$ is pure-injective, that is, if the sequence

$$0 \to \text{Hom}_R(C, G) \to \text{Hom}_R(B, G) \to \text{Hom}_R(A, G) \to 0$$

is exact for all pure exact sequences $0 \to A \to B \to C \to 0$ of $R$–modules. Now the notion of a pure exact sequence generalizes to the category of right $E$–modules for an arbitrary ring $E$. Namely, $G \to A \to B \to C \to 0$ is pure exact if and only if $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is exact for all left $E$–modules $M$.

Thus we may speak of pure-injective right $E$–modules. Clearly any injective module is pure-injective.

If $E$ is any ring and $C$ is an injective $E$–module, then $G$ is algebraically compact as an abelian group (1), page 176). More generally, if $R$ is a principal ideal domain, and $G$ is an $E$–bimodule that is injective as an $E$–module, then $G$ is algebraically compact as an $R$–module. We shall prove a stronger result.

Lemma 5. Let $E$ and $R$ be arbitrary rings, $D$ a pure-injective right $E$–module, and $X$ a left $R$–right $E$–module. Then $\text{Hom}_R(X, D)$ is a pure-injective right $R$–module.

Proof. Let

$$0 \to A \to B \to C \to 0$$

be a pure exact sequence of right $R$–modules. Then

$$0 \to A \otimes_R X \to B \otimes_R X \to C \otimes_R X \to 0$$

is an exact sequence of right $E$–modules. Moreover, (5) is pure, since tensoring (5) with any left $E$–module $M$ is the same as tensoring (4) with the left $R$–module $M \otimes_R E$.
Applying \( \text{Hom}_R(\cdot, D) \) to (5) gives the exact sequence
\[
0 \to \text{Hom}_R(G, \text{Hom}_R(X, D)) \to \text{Hom}_R(A \otimes_R X, D) \to \text{Hom}_R(X, D) \to 0,
\]
which is the same as
\[
0 \to \text{Hom}_R(G, \text{Hom}_R(X, D)) \to \text{Hom}_R(A \otimes_R X, D) \to \text{Hom}_R(A, \text{Hom}_R(X, D)) \to 0,
\]
and since (6) was an arbitrary pure exact sequence of right \( R \)-modules, this says that \( \text{Hom}_R(X, D) \) is a pure-injective right \( R \)-module.

**Corollary 1.** If \( E \) is any ring, then the underlying additive group of any pure-injective \( E \)-module is algebraically compact.

**Proof.** Let \( X = \mathbb{Z} \) and \( R = \mathbb{Z} \), in Lemma 5.

**Corollary 2.** If \( R \) is a principal ideal domain, and \( G \) is an \( R \)-module that is objective (or merely pure-injective) over \( E(G) \), then \( G \) is algebraically compact \( R \)-module.

**Proof.** Let \( X = E(G) \) with the natural \( R \)-module structure, and let \( G = D \), in Lemma 5.

We can further limit the structure of \( G \) without invoking the full force of objectivity. The key tool is the following lemma which says, roughly, that if \( G \) is pure-injective over \( E(G) \), then \( G \) cannot contain certain summands without containing the corresponding products.

**Lemma 6.** Let \( E \) be a ring and \( G \) a pure-injective \( E \)-module. If \( \{e_a \} \) is a family of orthogonal idempotents in \( E \), and if \( x_a \in e_a G \), then there is an \( x \in G \) such that \( e_a x = x_a \) for all \( a \).

**Proof.** Let \( L = \bigoplus e_a G \). Then \( L \) is a pure left ideal of \( E \), since \( L \) is a direct limit of pure submodules of \( E \). Now consider the map \( \phi : L \to G \) defined by \( \phi(e_a) = x_a \). This makes sense because \( x_a \in e_a G \), and the idempotents \( e_a \) are orthogonal. Since \( G \) is pure-injective, \( \phi \) extends to a map \( \phi : E \to G \), and \( x \in \phi(1) \) has the desired properties.

**Corollary.** Let \( R \) be a principal ideal domain and \( G \) an \( R \)-module. If \( G \)
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is pure-injective as a module over \( B(G) \), then \( G = R_p \cdot G \oplus D \), where \( G_p \) is a
finite direct sum of cyclic \( R_p \)-modules, and \( D \) is a finite \( R \)-module.

Proof. Since \( G \) is algebraically compact by Corollary 2 to Lemma 5, we can
write \( G = R_p \cdot G \oplus D \), where \( D \) is divisible, and \( G_p \) is the \( p \)-adic completion
of a direct sum of cyclic \( R_p \)-modules. Suppose \( G_p \) is the completion of \( \sum G_{p_i} \),
where \( G_{p_i} \) is a torsion cyclic \( R_p \)-module. Let \( \{ \kappa_p \} \) be the family of orthogonal
projections on the \( B_p \) that kill \( \prod G_{p_i} \cdot D \), and let \( \kappa_p \) be a generator of
\( B_p \). Then by Lemma 6, there is an \( \lambda \in G \) such that \( \kappa_p \lambda = \kappa_p \) for all \( \lambda \).
But that is impossible unless \( p \) is finite. A similar argument shows that \( D \) is of finite
rank.

4. Injectives

Throughout this section we assume that \( G \) is injective over \( B(G) \), and adopt
the notation of the last corollary. To complete the proof of the theorem, we must show
that one of the three listed conditions holds. This amounts to showing that certain
combinations of parameters are forbidden.

Lemma 7. Let \( R \) be a principal ideal domain and \( F \) a ring. Let \( M \) be an
\( E \)-\( R \)-bimodule that is injective as an \( E \)-module. If the \( q \)-primary \( R \)-submodule
of \( M \) is bounded, then any \( q \)-torsion-free \( R \)-module isomorphic to a\( q \)-torsion-free \( R \)-module is bounded, then any \( q \)-primary \( R \)-module is

Proof. Under the hypothesis, there exists an integer \( n \) such that \( q^n M \) is
\( q \)-torsion-free. Hence, multiplication by \( q \) induces an \( E \)-isomorphism from \( q^n M \)
onto \( q^n M \). Therefore, since \( M \) is injective, there is an \( E \)-endomorphism \( \phi \) of
\( M \) such that \( \phi \psi = \psi \) for \( \psi \) is the identity on \( q^n M \). If \( \theta \) is a projection onto a
\( q \)-torsion-free \( R \)-module isomorphic to \( M \), then \( \theta \) induces an isomorphism of \( F \)
such that \( \phi \theta = \theta \phi \) is the identity on \( q^n M \). Hence, since \( F \) is \( q \)-
torsion-free, \( \theta \) is the identity on \( F \), so \( F \) is \( q \)-divisible.

Corollary. If \( G_p \) is not torsion for some prime \( p \), then the \( q \)-primary
components of \( G \) are non-null.

Proof. If \( G_p \) is not torsion then \( G \) has a \( q \)-primary component isomorphic to \( R_p \). But
if the \( q \)-primary component of \( D \) is zero, then the \( q \)-primary submodule of \( G \)
is bounded, so \( R_p \) would be \( q \)-divisible by Lemma 7, a contradiction.

The following lemma plays the central role in eliminating the unwanted combinations:

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Lemma 8. Let $R$ be a principal ideal domain, and $G$ an $R$-module that is injective over its endomorphism ring $E$. Let $B$ be a nonzero divisible submodule of $G$, and let $S$ denote the endomorphism ring of $B$. Then if $A$ is a submodule of $G$, the natural map $A \to \text{Hom}_G(Hom_G(A, B), B)$ is onto. The same conclusion holds if $B$ is torsion-free divisible, and $G$ has no nonzero torsion divisible submodules.

Proof. In either case we can write $G = B \oplus H$, and there exist $e_1, e_2 \in B$ such that $\pi_1, \pi_2 \in S$ such that $e_1 = 0 \iff \ker \pi_1 = \ker \pi_2 \iff \pi_1 G \cap \pi_2 G$ is direct.

if $\lambda \in E$ and $\lambda(H) = 0$ then $\lambda = \sum \lambda_j e_j$, where $\lambda_j(H) = 0$ and $\lambda_j(G) \in B$. Let $\phi : \text{Hom}_G(A, B) \to B$ be the $S$-homomorphism and let $\lambda$ be the left ideal of $E$ generated by these endomorphisms $\lambda$ that kill some fixed complementary summand $C$ of $A$ and take $C$ into $B$. Define $\phi$ on $C$ by $\phi(\sum \lambda_j e_j) = \sum \phi(\lambda_j) e_j$, for all $e_j \in E$ and $c_j \in C = \phi(0), \phi(c_j) \subseteq B$. To show that this is an $S$-injective $E$-map, it suffices to show that $\sum \lambda_j c_j = 0$, then $\lambda_j(\phi(c_j)) = 0$. Suppose $\sum \lambda_j c_j = 0$. We may assume that $\lambda_j(0) = 0$, since $\phi(C)$ and $\phi(0)$ are in $B$. Hence $\lambda_j \in S$, $\phi(\lambda_j) = 0$, and $\lambda_j(H) = 0$ and $\lambda_j(G) \in B$. Note that we may view $\lambda_j$ as an element of $S$. Then $\lambda_j \phi(c_j) = 0$, so $\sum \lambda_j c_j = 0$, since $\sum \lambda_j c_j \subseteq \text{Hom}(A, B)$, and $\pi_2 \in S$, we have $\pi_2(\sum \lambda_j c_j) = \phi(0) = 0$, so $\sum \lambda_j c_j = 0$. Hence $\phi$ is $S$-injective.

Since $G$ is $E$-injective we can extend $\phi$ to an $E$-map $\phi : G \to B$. Let $\alpha = \phi(c)$. If $f \in E$, $f \alpha = 0$, and $f(G) \subset B$, then $f(e) = \phi(f(c)(e)) = \phi(c) = \phi(0)$, so evaluation at $\alpha$ reduces the map $\phi : \text{Hom}_G(A, E) \to B$, and we will examine the projection of $\alpha$ on $A$.

In order to use Lemma 8 we need to draw some of the consequences from our conclusion.

Lemma 9. Let $R$ be a principal ideal domain, and $B$ a rank-one divisible $R$-module with endomorphism ring $S$. If $A$ is a $R$-module of torsion-free rank $m$ on $A$, and $S$ is a natural map $A \to \text{Hom}_G(A, B)$, then $S$ is divisible. Moreover, this map is an $S$-homomorphism. Therefore if $A = \text{Hom}_G(A, B)$, $B$ is a submodule of $A$, $A$ is a submodule of $B$, and the natural map $A \to \text{Hom}_G(A, B)$, $B$ is onto. Then $A$ is a $S$-module. If $m$ is finite, the torsion-free $S$-rank of $A$ is finite and is the same as the rank of $A$.

Proof. Note that $S = B \oplus S \times K$, so $S$ is commutative. Let $F \subseteq C$ be a free $R$-module of rank $m$. Then $\text{Hom}_G(F, A)$ is a free $R$-module since $B$ is divisible. Moreover, this map is an $S$-homomorphism. Therefore if $A = \text{Hom}_G(F, B)$, $B$ is an $S$-submodule. Hence the torsion-free $S$-rank of $A$ is equal to $m$, since $S$ is a torsion-free $S$-rank of $N$ is equal to $m$, since $S$ is divisible. If $m$ is infinite, the torsion-free $S$-rank of $N$ exceeds $m$, a contradiction. If $m$ is finite, then the torsion-free $S$-rank of $N$ is equal to $m$, so the
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rank of $S$ must be 1. If $A$ is torsion-free, the map $A \to M$ is an $A$-module homomorphism, so the $A$-module structure of $M$ can be transferred to $A$.

We are indebted to Professor George Bergman for the proof of the next lemma. However, the lemma itself seems to be quite old, and a similar proof has been given by Mrs. Magdalena Ourosky. Kaplansky has some interesting comments on it in [2, page 82].

Lemma 10. If the rank of $P_q$ is 1, then $R$ is a complete discrete valuation ring.

Proof. Suppose $p$ is a prime in $A$, distinct from $q$. Then $p$ induces a valuation $v_p$ on $K$ and hence on $R_p$, $p$. Write $ap = b$ with $a, b \in K$. Let $x = \frac{a}{b} + p$. Then $v_p(x) > 0$. Let $\nu$ be a positive integer not divisible by the characteristic of $R/pR$, and not dividing $v_p(x)$. By Henrici's Lemma, $x$ has an $\nu^{th}$ root $\sqrt[\nu]{x}$ in $R$. But $v_p(x^{1/\nu}) = v_p(x)$ is not an integer. The lemma follows.

Corollary 1. If $G_q$ is not torsion, then $R$ is a complete discrete valuation ring.

Proof. If $G_q$ is not torsion, then $G$ has a subgroup $A$ isomorphic to $R_q$, and, by the Corollary to Lemma 7, a submodule $B$ isomorphic to $R_q$. $B$ is rank-one, by Lemma 9, $S = P_q$ has rank-one. $R$ is a complete discrete valuation ring.

Corollary 2. If $D \neq 0$, then $C_q = 0$ for all but finitely many $p$.

Proof. Let $A \neq C_q$. If $D \neq 0$, then, by Lemma 9, $G$ is rank-one, and the torsion-free rank of $A$ is finite, so $C_q = 0$ for all but finitely many $p$.

Corollary 3. If $D$ is mixed, then $R$ is a complete discrete valuation ring.

Proof. If $D$ is mixed then $G$ contains a subring $A$ isomorphic to $K$ and a submodule $B$ isomorphic to $R_q$. By Lemma 9, $S = P_q$ has rank-one, and by Lemma 10, $R$ is a complete discrete valuation ring.

The theorem is proved.

5. Questions and examples.

If $R$ is the ring of integers, then $R_q$ has infinite rank for all primes $q$. Hence the characterization of abelian groups that are injective over these endomorphism rings is
contained in conditions 1 and 2 of the theorem. If \( R \) is a complete discrete valuation ring, then we get additional injective from condition 3. Notice that in this case, \( R \) is an endomorphism ring, while \( R \) is not, which property is not affected under the taking of subrings.

Part of the proof concern itself with the possibility that \( R_{\mathfrak{p}} \) has finite rank bigger than 1. This cannot happen if \( K \) is a perfect field for then \( K \otimes R \) would be a finite separable extension of \( K \), and \( R_{\mathfrak{p}} \) would be the ring of integers over \( R_{\mathfrak{p}} \cap K \). Hence, \( R_{\mathfrak{p}} \) would have an integral basis, and \( x \) would be monomial to a direct sum of copies of \( R_{\mathfrak{p}} \cap K \), contradicting the fact that \( R_{\mathfrak{p}} \cap K \) is dense in \( R_{\mathfrak{p}} \). However, in the general case, \( R_{\mathfrak{p}} \) might have finite rank bigger than 1, as the following example shows.

Example. Let \( \mathbb{F} \) be a field of characteristic \( p \neq 0 \), such that \( [\mathbb{F} : \mathbb{F}^p] \) is infinite. Let \( \mathbb{F}(x) \) be the field of formal power series \( \sum_{n=0}^{\infty} a_n x^n \), such that \( a_n \) is an integer (not fixed) and \( a_0 \in \mathbb{F} \). Let \( \mathbb{K} \) be the subfield of \( \mathbb{F}(x) \) consisting of those power series whose coefficients generate a finite-dimensional extension of \( \mathbb{F} \). Choose \( \alpha \in \mathbb{F}(x) \mathbb{K} \), and let \( K = \mathbb{K} \cup \mathbb{F}(x) \alpha \) be a field maximal with respect to the property \( \forall \alpha \in K \). Then \( \mathbb{F}(x) \otimes \mathbb{K}(\alpha) \) has dimension \( p \) over \( K \), and if \( \mathcal{R} \) is the sheaf of \( K \)-valued power series with nonnegative exponents, then \( \mathcal{R} \) is a valuation ring with prime \( K \) and quotient field \( K \), whose completion, \( \mathbb{F}(x)\mathbb{K} \), has rank \( p \) over \( K \).

What abelian groups are pure-injective as modules over their endomorphism rings? What abelian groups are injective as modules over some ring? And the same questions for modules over an arbitrary principal ideal domain. Notice that Corollary 1 to Lemma 5 answers the question of what abelian groups are pure-injective over some ring, since algebraically compact groups are pure-injective over \( \mathbb{Z} \).

References