Normal Forms and Truth Tables for Interval-valued Fuzzy Logic

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Abstract

In this paper, we provide normal forms and truth tables for interval-valued fuzzy logic which are analogous to those for classical logic, that is, analogous to the disjunction and conjunction normal forms and truth tables for boolean algebras. We give an algorithm for rewriting an expression to obtain its disjunctive normal form. We also give an algorithm for obtaining the conjunctive normal form of an expression from its table of truth values.

1. Interval-Valued Fuzzy Sets

If $X$ is the universal set, then interval-valued fuzzy sets are functions from $X$ to the set of subintervals of the unit interval $[0, 1]$. All of the computations done with these intervals depend only on the endpoints. Thus we will identify the interval $[a, b] \subseteq [0, 1]$ with the pair $(a, b)$ of its endpoints. We use the standard lattice theory notation

$$[0, 1] = \{(a, b) : a, b \in [0, 1], a \leq b\}$$

for the set of all such pairs, so interval-valued fuzzy sets are functions $A : X \rightarrow [0, 1]$. For the algebra of interval-valued fuzzy sets, we take the set $[0, 1]$ together with componentwise operations coming from the operations on $[0, 1]$. We define

$$(a, b) \land (c, d) = (a \land c, a \land d)$$

$$(a, b) \lor (c, d) = (a \lor c, b \lor d)$$

which give the usual lattice min and max, or "and" and "or" operations.

Setting $I = ([0, 1], \land, \lor, 0, 1)$, the resulting structure has the standard notation

$$\mathcal{I} = ([0, 1], \land, \lor, 0, 1)$$

That is, $\mathcal{I}$ is the set $[0, 1]$ with componentwise operations. This is a fundamental lattice theoretical construction: from a lattice $L$, form $L^{\mathcal{I}} = \{(a, b) : a, b \in L, a \leq b\}$ and use componentwise operations. The resulting lattice has many of the same properties as the original lattice. In particular, if $L$ is a complete distributive lattice, such as $\mathbb{I}$, then so is $L^{\mathcal{I}}$. The usual negation, or complement, on $\mathcal{I}$ is given by

$$(a, b)' = (b', a')$$

where $x' = 1 - x$. With this operation, $\mathcal{I}$ becomes a De Morgan algebra

$$(\mathcal{I}, ', \land, \lor, 0, 1)$$

or is, $(\mathcal{I}, ', \land, \lor, 0, 1)$ is a distributive lattice satisfying the De Morgan laws:

$$(a \land b)' = a' \lor b'$$

$$(a \lor b)' = a' \land b'$$

This in turn yields a De Morgan algebra on the set $\mathcal{F}(X)$ of all interval-valued fuzzy sets with the operations

$$(A \land B)(x) = A(x) \land B(x)$$

$$(A \lor B)(x) = A(x) \lor B(x)$$

$$(A')(x) = (A(x))'$$

So, $\mathcal{F}(X) = ([0, 1], \land, \lor, 0, 1)$. It is this De Morgan algebra with which we will be concerned. These connectives are the classical AND, OR, and NOT connectives for interval-valued fuzzy sets. We will be concerned with normal forms for expressions built up from interval-valued fuzzy sets and these connectives.

2. What are Normal Forms?

The notion of a normal form, or canonical form, provides mathematicians. This is the identification of a unique selection from each equivalence class of an equivalence relation together with a method of finding that selection, giving an arbitrary element of an equivalence class. To
with \( \land \) and \( \lor \) the lattice operations, and \( \neg \) the operation that interchanges \( 0 \) and \( 1 \) and \( \land \) and \( \lor \). This algebra is called a distributive lattice. A distributive lattice satisfies De Morgan's laws, which state that for any variables \( x, y, z \):

\[

\neg(x \land y) = \neg x \lor \neg y \\
\neg(x \lor y) = \neg x \land \neg y
\]

3. Normal Forms for Interval-valued Fuzzy Logic

The problem is this. The polynomials, or terms over \( B = \{0, 1\} \), are expressions in a finite number of variables \( x, y, z, \ldots \), with the connectives \( \land, \lor \), and \( \neg \). Such terms may be combined using these connectives, and this yields a term in \( B \), called the term algebra. Two such polynomials \( p \) and \( q \) are equivalent over \( B \) if and only if \( p(b, c, d) = q(b, c, d) \) for all \( b, c, d \in B \). This is logical equivalence for two such terms with the algebra of truth values \( B \). This equivalence relation is a congruence, equating the equivalence classes to themselves by combining the given connectives. These equivalence classes form a De Morgan algebra which we denote \( \mathcal{L} \). We want to choose from each equivalence class an element (is normal form) so that its form informs whether or not it is one of the chosen elements. Further, given any term, we should give an algorithm for transforming it to that normal form.

The basis for getting normal forms for the De Morgan logic \( \mathcal{L} \) is the notion of join-irreducible elements in lattices, and determining those elements in \( \mathcal{L} \).

Definition 1. An element \( z \) of a finite lattice \( \{0, 1, \ldots, n\} \) is join-irreducible if it is distinct from 0 and is not the join of any smaller elements. Explicitly, an element is meet-irreducible if it is distinct from 0 and is not the meet of any smaller elements.

In \( \mathcal{L} \), every element except 0 and 1 is join-irreducible. This will make the problem of identifying normal forms rather easy. The pertinent fact for us is the following proposition.

Proposition 2. In a finite distributive lattice, every element is uniquely the join of pairwise incomparable join-irreducible elements.

We are assuming that the set \( V \) of variables is finite. Every element \( e \in \mathcal{L} \), that is, every term, is an expression in some finite number of elements of \( V \). Now, use the axioms of a De Morgan algebra to transform it into a join of meets of the variables and their negations. The element is now in lattice type. The variables and their negations are called literals. Thus, the result is an el-

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element of \( T \) of the form \( u_1 \lor v_2 \lor \cdots \lor a_m \), with the \( a_i \) distinct and each \( a_i \), a meet of distinct literals. This transformation of \( T \) does not change the element of \( L_T \) that represents. That is, \( T \land u_1 \lor v_2 \lor \cdots \lor a_m \) are logically equivalent. But there are only finitely many elements in \( T \) of the form, there being only finitely many variables. Thus, if the set of variables is finite, then \( L_T \) is finite.

We have observed that every element of \( L_T \) is a join of meets of literals. So for an element of \( L_T \), to be join irreducible, it must be a meet of literals. The next proposition follows from these observations and Proposition 2.

Proposition 3. Every element of \( L_T \) is uniquely the join of pairwise incomparable join-irreducibles, and every join-irreducible is a meet of literals.

The real bear of the matter is to identify the meets (conjunctions) of literals that are join irreducible, and the ordering between these join irreducibles.

Theorem 4. Let \( a \) and \( b \) be conjunctions of distinct literals in \( L_T \) with \( b \subseteq a \). Then

1. Every literal that occurs in \( a \) also occurs in \( b \).
2. \( a = b \) if and only if they are the conjunctions of exactly the same literals.
3. \( a \) is join irreducible.

Example 5. Let \( a = z_1 \land z_2 \land z_3 \) and \( b = z_1 \land z_2 \land z_4 \). Then \( b \subseteq a \), \( a \neq b \), and \( a \lor b \lor a' \) are both join irreducible.

This theorem tells us what the join irreducibles are, and tells us what the ordering is between these join irreducibles. It is clear that these join irreducibles form a lattice. The set of join irreducibles is not a sublattice of \( L_T \) but its lattice structure is induced from the partial ordering of \( L_T \).

The normal form for the De Morgan algebra \( L_T \) now follows readily. The join irreducible elements are exactly the conjunctions of distinct literals, and the element 1. Since \( L_T \) is a finite distributive lattice, every element can be written uniquely as a conjunction of pairwise incomparable join irreducibles.

Following is a precise procedure for putting an arbitrary term \( t \) to the variables \( x_1, \ldots, x_n \) in this De Morgan disjunctive normal form.

1. Given a term \( t \) in \( T \), first use De Morgan's law to move all the negations in, so that the term is rewritten as a term \( t' \), which is of lattice type in the literals, \( 0 \) and \( 1 \).
2. Next use the disjunctive law to obtain a new term \( t'' \) from \( t' \), which is a disjunction of conjunctions involving the literals, \( 0 \), and \( 1 \). At this point, discard any conjunction in which \( 0 \) or \( 1 \) appears as one of the conjuncts. Also discard any repetition of literals from any conjunction, as well as \( 1 \) and \( 1' \) from any conjunction in which they do not appear alone of a conjunction consists entirely of \( 1 \)'s and \( 1' \)s, then replace the whole conjunction by 1.

If no restrictions are left, then you have the empty join—normal form for \( t \). This yields a term \( w \).

3. Now discard all non-maximal conjunctions among the conjunctions that \( w \) is a disjunction of. The type of conjunctions we now are dealing with are either conjunctions of literals or \( 1 \) by itself. This process yields a term \( u \).

The terms thus obtained in row in De Morgan disjunctive normal form, and represents the same element as \( u \) when interpreted in \( L_T \).

The dual forms to these disjunctive normal forms are called conjunctive normal forms. For example, every element is uniquely the conjunction of pairwise incomparable disjunctions of literals, and any disjunction of literals is most irreducible. We do not go through the details for these conjunctive normal forms. We just remark that applying De Morgan laws to the negation of the disjunctive normal form gives the conjunctive normal form of that negation.

4. Truth Tables for Interval-Valued Fuzzy Logic

The algebra \( L_T \) for two variables \( x, y \) has the truth values shown in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x \lor y )</th>
<th>( x \land y )</th>
<th>( x' )</th>
<th>( y' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Two expressions in \( L_T \) are equivalent if and only if they produce a table with the same truth values. \( \wedge \) is a simple
exercise to derive the disjunctive normal form and the conjunctional normal form for expressions in Boolean algebra from "truth tables," constructed from the relevant algebra of truth values. The disjunctive and conjunctional normal forms for the interval-valued fuzzy case can also be recovered from truth tables. We shall show how the disjunctive normal form can be constructed from truth tables.

Disjunctive normal forms are gotten from a truth table as described below. The following table gives some sample rows with four variables.

<table>
<thead>
<tr>
<th>r₁</th>
<th>r₂</th>
<th>r₃</th>
<th>r₄</th>
<th>t</th>
<th>pick up</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>u</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>x₁ \wedge x₂ \wedge x₃</td>
</tr>
<tr>
<td>1</td>
<td>u</td>
<td>0</td>
<td>v</td>
<td>1</td>
<td>x₁ \wedge x₂ \wedge x₄</td>
</tr>
<tr>
<td>1</td>
<td>v</td>
<td>0</td>
<td>u</td>
<td>1</td>
<td>x₁ \wedge x₂ \wedge x₃ \wedge x₄</td>
</tr>
<tr>
<td>1</td>
<td>v</td>
<td>u</td>
<td>0</td>
<td>1</td>
<td>x₁ \wedge x₂ \wedge x₃ \wedge x₄</td>
</tr>
</tbody>
</table>

- For those rows that have value 1 in the column of the expression t to be put into disjunctive normal form, and truth values 0 or 1 and possibly u or v (but not both) for the variables, form the conjunction of the variables with truth value equal to 1 with the negations of the variables with truth value equal to 0 (and leave out variables with truth value equal to u or v). If the row has values all u's (or all v's) pick up p = 1.

- For those rows that have value 1 in the column of the expression t to be put into disjunctive normal form, and both u and v occur as truth values for the variables, form two conjunctions: one using the variables with truth value equal to 1 and the negations of the variables with truth value equal to 0, and using both the variables and the negated variables of the variables with truth value equal to 0 and using both the variables and the negated variables of the variables with truth value equal to 1; and using both the variables and the negated variables of the variables with truth value equal to 1. For those rows that have value u in the column of the expression t to be put into disjunctive normal form, form the conjunction using the variables with truth value equal to 1, the negations of the variables with truth value equal to 1, the negations of the variables with truth value equal to 0, mapping both the variables and the negated variables of the variables with truth value equal to v (and leave out variables with truth value equal to u).

- Ignore those rows that have value 0 in the column of the expression t to be put into disjunctive normal form.

The irredundant De Morgan-disjunctive normal form for the expression is then obtained by discarding redundant conjunctions—that is, any conjunction that contains the same, and possibly more, literals in another conjunction in the term—and then taking the disjunction of the conjunctions in the last column.

5. The De Morgan Logics

It is natural to ask about the number of disjunctive normal forms, or equivalently, the size of the algebra Lₚ, or equivalently, the number of truth tables. This number depends, of course, on the number of variables. In the two-variable case, with Boolean logic there are exactly 16 disjunctive normal forms, and in the n-variable case, there are 2ⁿ⁺¹ disjunctive normal forms. In the two-variable case, with interval-valued fuzzy logic (De Morgan logic), there are 168 disjunctive normal forms. We do not know a formula for the n-variable case, but the number increases rapidly with the number of variables.

One way to count the number of elements of the De Morgan logic is to compare it with the Boolean logic via the homomorphism π : Lₚ → L₂ that interprets a De Morgan polynomial as a Boolean polynomial. The congruence classes induced by this homomorphism are intervals of L₂. For the two-variable case, the number of elements that map to a given Boolean element is shown in the following table.

<table>
<thead>
<tr>
<th>Boolean short form</th>
<th># De Morgan forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>x \lor y</td>
<td>26</td>
</tr>
<tr>
<td>x \lor y'</td>
<td>9</td>
</tr>
<tr>
<td>x' \lor y</td>
<td>9</td>
</tr>
<tr>
<td>x' \lor y'</td>
<td>9</td>
</tr>
<tr>
<td>x \land y</td>
<td>9</td>
</tr>
<tr>
<td>x \land y'</td>
<td>9</td>
</tr>
<tr>
<td>x' \land y</td>
<td>9</td>
</tr>
<tr>
<td>x' \land y'</td>
<td>9</td>
</tr>
<tr>
<td>x \lor y' \lor x'</td>
<td>4</td>
</tr>
<tr>
<td>x \lor y \lor x'</td>
<td>4</td>
</tr>
</tbody>
</table>

TOTAL 168

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If a term is placed in its Boolean disjunctive normal form $\phi$ and also in its Boolean conjunctive normal form $\psi$, and these two terms are interpreted in the De Morgan logic, it is the case that $\phi \leq \psi$ and $(\phi \land \psi) = \delta$ if and only if all of the conjunction classes are obtained in this way. Thus counting the number of elements in $\mathcal{L}_{\phi}$ comes down to counting the number of normal forms in each interval $[d, e]$. In general, there is an inequality $2^{\leq d} \leq \# \text{De Morgan forms} \leq 4^{d}$, but as you can see from the previous table, and $4^{d} = 4294967296$, this is a very poor upper bound! These numbers have been compiled by J. Berman and others [1, 2, 3, 4] for several small values of $d$.

6. Conclusions

There are disjunctive and conjunctive normal forms and truth tables for interval-valued fuzzy (De Morgan) logic analogous to those for classical (Boolean) logic and there are algorithms for obtaining these normal forms. Any disjunction of conjunctions that does not contain redundancies is in disjunctive De Morgan normal form. Similarly, any conjunction of disjunctions that does not contain redundancies is in conjunctive De Morgan normal form.

References


