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PROJECIVE CLASSES OF ABELIAN GROUPS

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Let \( A \) be an Abelian category. Associated with any class \( C \) of objects of
\( A \) is the class \( E(C) \) of \( C \)-pure, or proper, short exact sequences. A short
exact sequence \( A \rightarrow B \rightarrow C \) is in \( E(C) \) if and only if the sequence
\[ \text{Hom}(A, i) \rightarrow \text{Hom}(B, i) \rightarrow \text{Hom}(C, i) \]
is exact for all objects \( i \) in \( C \). Also, given any class \( D \) of short exact
sequences, there is associated the class \( F(D) \) of objects \( C \) for which \((4)\)
is exact for all sequences \( A \rightarrow B \rightarrow C \) in \( D \). It is well known that
\[ E(V(E(C))) = E(C) \]
and
\[ F(V(F(D))) = F(D) \]
for any classes \( C \) and \( D \) of objects and short exact sequences, respectively.

The class \( P(E(C)) \) is the projective closure of \( C \), and in case
\( C = \mathbb{P} \), \( C \) is called a projective class. Objects in \( \mathbb{P} \) are called \( C \)-projective.
Let \( C^p \) denote the class of all direct sums of direct sums of objects in \( C \).
Clearly \( C^p \subseteq \mathbb{P} \) is always the case. Theorem 1 gives an elementary but useful
criterion for \( C^p \) to be equal to \( \mathbb{P} \) and to contain enough projectives.

Theorem 2 describes a condition on \( C \) which implies \( \mathbb{P} \) contains divisible
groups. Using these theorems we are able to describe the projective closure of
the class of torsion complete Abelian p-groups, namely, it is the class of groups
of the form \( C \otimes D \) where \( C \) is a direct sum of torsion complete p-groups, \( F \)
is a free Abelian group, and \( D \) is a divisible p-group. Also the closure of the
class of reduced torsion free groups is described, being the class of all
torsion free groups.

Finally, several theorems are given which relate the existence of divisible
groups in \( \mathbb{P} \) to properties of the short exact sequences in \( E(C) \).

1. PROJECTIVE CLASSES.

For a class \( C \) of objects, let \( C^p \) denote the class of direct sums of
objects of \( C \), and let \( C^s \) denote the class of direct sums of objects of \( C^p \).

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Then $C \subseteq C' \subseteq \text{Coclass } C \subseteq C''$, so $C''$ is the first candidate when one looks for a description of $C'$. (The question of whether $C' \subseteq C''$ often arises, and it seems to be particularly reasonable when $C'''$ is a closed projective class.)

The following theorem gives a necessary and sufficient condition for $C''$ to be a projective class with enough projectives.

**Theorem 1.** Let $A$ be an abelian category with infinite direct sums, and $C$ a class of objects of $A$. Then $C' \subseteq \text{Coclass } C \subseteq C''$, and contains enough projectives if and only if for each $A$ in $A$ there exists an epimorphism $C \rightarrow A$ with $C$ in $C'$ and a set $\mathcal{S}(A) \subseteq C''$ such that every homomorphism $P \rightarrow A$ with $P$ in $C$ can be factored through an object in $\mathcal{S}(A)$.

**Proof.** Suppose $C' \subseteq C \subseteq \text{Coclass } C \subseteq C''$ contains enough projectives, then there exists a sequence $0 \rightarrow C \rightarrow A$ in $E(C)$ with $C$ in $C'$. This gives the desired epimorphism, and the set $\mathcal{S}(A) = \{C\}$ satisfies the statement of the theorem.

Conversely, suppose $C \rightarrow A$ is an epimorphism with $C$ in $C'$, and $\mathcal{S}(A)$ satisfies the statement in the theorem. Let $G = \bigoplus \sum \mathcal{H}(C) \otimes \mathcal{S}(A) \otimes \mathcal{S}(A)$, and let $r : G \rightarrow A$ be the direct sum of the maps from the index sets not the epimorphism $C \rightarrow A$. Then $r$ is an epimorphism, and $r$ is in $C'$. We will show that the sequence $0 \rightarrow C \rightarrow G \rightarrow A$ belongs to $E(C)$. To do this it suffices to show that $\text{Hom}(C', r) \otimes \mathcal{H}(C) \rightarrow \text{Hom}(C, r) \otimes \mathcal{H}(C)$ is an epimorphism for each $P$ in $C'$. Let $s : P \rightarrow A$. Then there exists an $s$ in $\mathcal{S}(A)$ and maps $P \rightarrow s$ such that $\text{Hom}(C', r) \otimes \mathcal{H}(C) \rightarrow \text{Hom}(C, r) \otimes \mathcal{H}(C)$ is an epimorphism for each $P$ in $C'$.

There are two common and rather trivial cases in which the hypotheses of the theorem are satisfied, which provide us with numerous examples of projective classes.

1. $C$ is a set, and for each $A$ in $A$ there is an epimorphism $C \rightarrow A$ with $C$ in $C'$.

2. $C$ is closed under homomorphic images, and for each $A$ in $A$ there is an epimorphism $C \rightarrow A$ with $C$ in $C'$ (see (6)).

In case I, take $\mathcal{S}(A) = C$ for all $A$. In case II, take $\mathcal{S}(A) = \{B \subseteq A | B \in C\}$.

Some examples may be considered in case I for the category of abelian groups are the set of cyclic groups, the set of countable groups, the set of free abelian groups, free abelian groups, the set containing $\mathcal{S}$ and all rank-one divisible
groups, the set of all reduced rank one groups, and the set of all reduced rank one torsion free groups. For each of the sets \( C \) of the last sentence, it is known that \( C = C^c \), so the projective classes are respectively, the direct sum of cyclic groups, the direct sums of countable groups, the free groups, the project sums of rank one (i.e. completely decomposable) torsion free groups, the groups which are a direct sum of a free with a divisible, the completely decomposable reduced groups, and the completely decomposable reduced torsion free groups.

In general, knowing that \( \mathbb{Z} = C^c \) is still an incomplete description of \( \mathbb{Z} \).

For example, the closure of the set of finite rank torsion free groups, although it is the class of direct summands of direct sums of finite rank torsion free groups, contains indecomposable torsion free groups of infinite rank [A.I.G. Dorn, oral communication].

For the category of Abelian p-groups, the set of cyclic p-groups and the set of torsion complete p-groups of cardinality \( \aleph_0 \) provide a couple of examples of Case I. Here again it is known that \( C^c = C^c [\mathbb{Z}] \).

The class of Abelian groups consisting of \( \mathbb{Z} \), together with all of the torsion groups is an example of Case II, but not Case I. By Theorem 1, the class of groups of the form \( F \otimes T \), with \( F \) free and \( T \) torsion, is a projective class with enough projectives.

2. Projective Classes Which Contain Divisible Groups.

The following theorem contains a criterion for the projective closure of a class of Abelian groups to contain certain divisible groups. The symbol \( \rightarrow \) means torsion free group.

**Theorem 2.** Let \( C \) be a class of Abelian groups and \( p \) a prime or \( \infty \). If for each cardinal \( \kappa \), there exists a p-group \( G \) and \( C \) with a subgroup \( H \) such that \( \leq \kappa \), \( \leq \kappa \), \( \leq \kappa \), and \( \frac{G}{H} \) is divisible, then \( \mathbb{Z} \) contains the divisible p-groups.

**Proof.** Let \( A \rightarrow B \rightarrow C \) be a sequence in \( E(C) \), with \( C \) a rank one divisible p-group. In order to show that \( C \) is in \( \mathbb{Z} \) it suffices to show that every such sequence is splitting exact. For if \( A \rightarrow X \rightarrow f \) is any sequence in \( E(C) \), the sequence \( \text{Hom}(C; X) \rightarrow \text{Hom}(C; Y) \rightarrow \text{Ext}(C; A) \) is exact, and element in the image of \( f \) are represented by sequences of the form \( A \rightarrow B \rightarrow C \) in \( E(C) \).

We first show that \( B \) cannot be a reduced group. It may be assumed that \( A \) is reduced, since the sequence \( A/AA \rightarrow B/AA \rightarrow C \) still belongs to \( E(C) \).

Choose a p-group \( G \) in \( C \) with a subgroup \( H \) such that \( \leq \kappa \), \( \leq \kappa \), and \( \frac{G}{H} \) is divisible. We may assume \( G \) is reduced, as otherwise
we have the conclusion of the theorem already. We may also assume $G/H$ is a $p$-group, and that isomorphic to a direct sum of copies of $C$. Since $G$ is in $C$, the map $\text{Hom}(G, B) \rightarrow \text{Hom}(G, C)$ is an epimorphism, implying that $[\text{Hom}(G, B)] \subseteq [\text{Hom}(G, C)]$, now if $G$ is reduced, since $G/H$ is divisible, the sequence $0 \rightarrow \text{Hom}(G, B) \rightarrow \text{Hom}(G, H)$ is exact, so $[\text{Hom}(G, B)] \subseteq [\text{Hom}(G, H)] < [G/H] < [G]$. But since $G$ is infinite, $[\text{Hom}(G, C)] = [G]$. Thus $B$ cannot be reduced.

Let $B$ be the maximal divisible subgroup of $B$. Now $B \neq 0$, since $A$ is reduced and $D \neq 0$, so $(D)A/B$ is a non-zero divisible subgroup of $B/H$, which is a rank one divisible group. It follows that $(D)A/B$. Now $D \cap A = 0$, whence the sequence splits as desired. Let $B = D \times A$. Then we have the exact sequence

$\text{Hom}(G, B) \rightarrow \text{Hom}(G, A) \rightarrow \text{Hom}(G, C) \rightarrow 0$.

Set $[\text{Hom}(G, B)] < [G]$. Since $B$ is reduced, thus the index of $\text{Hom}(G, D)$ in $\text{Hom}(G, C)$ must be less than $[G]$. The exact sequence $\text{Hom}(G, B) \rightarrow \text{Hom}(G, C) \rightarrow \text{Ext}(G/B, \mathbb{Z}) \rightarrow 0$ shows we must have $[\text{Hom}(G, D)] = [\text{Hom}(G, C)] < [G]$. Also the sequence $\text{Hom}(G/B, \mathbb{Z}) \rightarrow \text{Ext}(G, M_B \otimes \mathbb{Z}) \rightarrow 0$ shows we must have $[\text{Hom}(G/B, \mathbb{Z})] < [G]$. Now since $G/B$ is divisible, the sequence $\text{Hom}(G/B, \mathbb{Z}) \rightarrow \text{Ext}(G, M_B \otimes \mathbb{Z}) \rightarrow 0$ shows we must have $[\text{Hom}(G/B, \mathbb{Z})] < [G]$. Hence the sequence $\text{Ext}(G/B, \mathbb{Z}) \rightarrow \text{Ext}(G, M_B \otimes \mathbb{Z}) \rightarrow 0$ shows we must have $[\text{Ext}(G/B, \mathbb{Z})] < [G]$. Hence $\text{Ext}(G/B, \mathbb{Z}) = 0$, so the sequence $\text{Ext}(G/B, \mathbb{Z}) \rightarrow \text{Ext}(G, M_B \otimes \mathbb{Z}) \rightarrow 0$ shows we must have $[\text{Ext}(G/B, \mathbb{Z})] = 0$. Since this implies $D \cap A$ is divisible, it must be 0.

We use this theorem to describe the closure of the classes of Abelian groups which do not satisfy either Case I or Case II of the previous section, the class of all pro-primary torsion complete groups and the class of all torsion-free reduced groups. Partial results are given in the following two Corollaries.

**Corollary 1.** If a pro-projective class contains all pro-primary torsion complete groups then it contains the pro-primary divisible groups.

**Proof.** For any cardinal $\kappa$ let $\mathfrak{a}_\kappa$ be a cardinal such that $\mathfrak{a}_\kappa = 2^\kappa$. Let $G = \mathfrak{a}_\kappa$ and set $N$ equal to a direct sum of copies of $G$. Let $G$ be the torsion completion of $N$, then $G/H$ is divisible and $[C] = [H]$. Thus we have $[N] = 2^\kappa$ and $[G/H] < 2^{\mathfrak{a}_\kappa}$.

**Corollary 2.** If a pro-projective class contains all reduced torsion-free groups then it contains all torsion-free groups.

**Proof.** For any cardinal $\kappa$, let $\mathfrak{a}_\kappa$ be a cardinal such that $\mathfrak{a}_\kappa = 2^\kappa$. Set $H$ equal to a direct sum of copies of the integers $\mathbb{Z}$, let $G = \mathfrak{a}_\kappa \mathbb{Z}$, which is torsion-free and reduced. If $I$ is the divisible envelope of $H$, the
isomorphism \( \text{Hom}(\mathbb{Z}/n\mathbb{Z}, D/R) \cong \text{Ext}(\mathbb{Q}/n\mathbb{Z}, G) \) shows the cardinalities work out right.

The existence of cardinal numbers such as the \( a \)'s chosen in these two corollaries follows easily from the generalized continuum hypothesis. We include the following proposition to release us from that restriction.

**PROPOSITION.** If \( b \) is any cardinal number then there exists a cardinal number \( a > b \) such that \( 2^a = 2^b \).

**PROOF.** Set \( a = \alpha_1 + \alpha_2^2 + \alpha_3^3 + \ldots \). Then \( 2^a = 2^{\alpha_1} \cdot 2^{\alpha_2^2} \cdot 2^{\alpha_3^3} \cdot \ldots \). Hence \( 2^b \leq 2^a \).

On the other hand if \( f \) is a function from the integers to \( a \) we can map \( f \) to its range. Any countable subset of \( f \) is the image of at most \( 2^{\alpha_0} \) functions \( \alpha \), hence \( 2^a = 2^{\alpha_1} \cdot 2^{\alpha_2^2} \cdot 2^{\alpha_3^3} \cdot \ldots \).

The descriptions of the projective closures of the class of \( p \)-primary torsion complete groups and the reduced torsion free groups are completed in the following two theorems. For Theorem 3 we need the following lemma.

**LEMMA.** Let \( R \) be a reduced \( p \)-group and \( C \) a torsion complete \( p \)-group. Then every homomorphism from \( C \) to \( R \) can be factored through some torsion complete \( p \)-group of cardinal \( \leq |R|^{|C|} \).

**PROOF.** Let \( h : C \rightarrow R \) and let \( B = \sum h(b_1) \) be a basic subgroup of \( C \).

For \( i \in I \), choose a \( b_i(a) \) having smallest order such that \( h(b_i(a)) = h(b_i) \) and that \( J = \{ a(a) \mid J \} \) satisfies \(|J| \leq |R|\). Then \( A = \sum h(b_1) \) is a summand of \( B \).

The function \( s : I \rightarrow J \) determines a homomorphism \( k : B \rightarrow A \) for which \( h = \text{ker} k \). Extend \( k \) to a homomorphism \( \tau : C \rightarrow A \), and we still have \( h = \text{ker} \tau \).

Now \( |C| \leq |R|^{|C|} \), so we have factored \( h \) through a \( p \)-primary torsion complete group of cardinal \( \leq |R|^{|C|} \).

**THEOREM 3.** Let \( C \) be the class of \( p \)-primary torsion complete groups. Then the projective closure of \( C \) consists of all groups of the form \( F \times C \), where \( F \) is free, \( C \) is a direct sum of \( p \)-primary torsion complete groups and \( F \) is \( p \)-primary divisible.

**PROOF.** Let \( D \) be the class consisting of \( \pi(p) \) and all of the \( p \)-primary torsion complete groups. Now \( X \) is in the projective closure of any class of Abelian groups, and by Corollary 1, \( \pi(p) \) is in \( T \) as well. Thus \( X \subseteq B \). We show that \( D \) satisfies the hypothesis of Theorem 1, so that \( T \subseteq B \).

If \( G \) is an Abelian group, write \( G_{p} = D \cdot B \), where \( G_{p} \) is the \( p \)-primary component of \( G \). \( D \) is divisible and \( B \) is reduced. Let \( S(G) \) be the set consisting of \( \pi(p) \) and all groups \( D \cdot A \) where \( A \) is a \( p \)-primary torsion complete group of cardinal \( \leq |B|^{|B|} \). Let \( f : C \rightarrow D \), with \( C \) a \( p \)-primary
torsion complete groups. Then the image of \( f \) is contained in \( G_p \). Let \( g : C \rightarrow B \) and \( h : B \rightarrow A \) be the maps obtained by following \( f \) by the projections, so \( f = g \circ h \). Using the lemma, let \( x \in B \) with the domain \( T \) of \( B \) a torsion complete \( p \)-group of cardinal \( \leq 1 \). Then the composition \( g \in \text{Aut}(p^A) \) factors through a group in \( S(G) \) if

This implies, by Theorem 1, that \( T \in \mathfrak{C}_i \). Thus the groups in \( T \) are

sums of direct sums of copies of the integers, \( \mathbb{Z}(p^n) \) and \( p \)-primary torsion complete groups. It is well known that such groups are of the form \( \mathbb{Z} \oplus C \) where \( \mathbb{Z} \) is a direct sum of copies of \( \mathbb{Z}(p) \), and \( C \) is a sum of a direct sum of \( p \)-primary torsion complete groups.

Thus \( C \) is actually a direct sum of \( p \)-primary torsion complete groups is a theorem of F. Will [2].

**Theorem 4.** The projective closure of the class of reduced torsion free groups is the class of all torsion free groups.

**Proof.** By Corollary 3, the projective closure of the class of reduced torsion free groups contains all of the torsion free groups, so we need only show that this class is projectively closed. Let \( P \) be the group of \( p \)-adic integers. The exact sequence \( 0 \rightarrow \mathbb{Z}(p^n) \rightarrow \mathbb{Z}(p^n) \rightarrow 0 \) yields an exact sequence \( \text{Hom}(\mathbb{Z}(p^n), \mathbb{Z}(p^n)) \rightarrow \text{Hom}(\mathbb{Z}(p^n), \mathbb{Z}(p^n)) \rightarrow 0 \) for every torsion free group \( G \), where \( \phi \) is an isomorphism. However, if \( T \) is a reduced torsion group the map \( \text{Hom}(\mathbb{Z}(p^n), \mathbb{Z}(p^n)) \rightarrow \text{Hom}(\mathbb{Z}(p^n), \mathbb{Z}(p^n)) \phi \) is one-to-one and \( \text{Hom}(\mathbb{Z}(p^n), \mathbb{Z}(p^n)) \phi \neq 0 \) so \( T \) cannot be in the projective closure of the class of torsion free groups. The sequence \( \text{Cl}(p) \rightarrow \mathbb{Z}(p^n) \rightarrow \mathbb{Z}(p^n) \) yields the exact sequence \( \text{Hom}(\mathbb{Z}(p^n), \mathbb{Z}(p^n)) \rightarrow \text{Hom}(\mathbb{Z}(p^n), \mathbb{Z}(p^n)) \rightarrow 0 \) for every torsion free group \( G \), but it does not split, so \( T(p) \) cannot be in the projective closure of the torsion free groups. It follows that the projective closure of the torsion free groups contains only torsion free groups.

More information about this projective class is contained in [3] where it is shown, among other things, that the class of torsion free groups contains enough projectives.

3. **Proper Exact Sequences for which Divisible Groups are Projective.**

For certain classes \( C \) of Abelian groups it is possible to describe the exact sequences in \( E(C) \). For example, for the set of cyclic groups, \( E(C) \) is the class of pure exact sequences; i.e., \( 0 \rightarrow G \rightarrow G \rightarrow 0 \) is in \( E(C) \) if and only if \( A \cap B = 0 \) for all positive integers \( n \). J. R. Burke [1] has given a characterization for \( C \), all reduced countable \( p \)-groups. Namely,
A \rightarrow B \rightarrow C is in \mathcal{E}(C) if and only if
\[ (p^{\infty}(C))_p = (A[p^{\infty}](p))/A \]
for all \( p \in \mathbb{P} \), the first uncountable ordinal. If \( C \) is the class of all rank one\(^*\) Abelian groups and \( A \rightarrow B \rightarrow C \) is an exact sequence of torsion-free groups, the sequence is in \( \mathcal{E}(C) \) if and only if every count of \( B \) and \( A \) is regular, i.e., contains an element having minimal height. This follows from the theorem of Kaplansky [1, page 150]. If \( A \rightarrow B \rightarrow C \) is a sequence of torsion groups, it belongs to \( \mathcal{E}(C) \) if and only if \( A \) is pure in \( B \) and the sequence \( \delta A \rightarrow B \rightarrow \delta C \) of divisible subgroups is exact. This is an easy consequence of Theorem 6 of this paper. The mixed case, for \( C \) the set of rank one groups, has not yet yielded to description.

The following theorem describes the classes \( \mathcal{E}(C) \) for which \( \mathbb{T} \) contains divisible groups.

**Theorem 5.** The group \( \mathbb{T} \) of rationals is in \( \mathbb{T} \) if and only if for each sequence \( A \rightarrow B \rightarrow C \) in \( \mathbb{E}(C) \), the sequence of subgroups
\[ A \cap \delta B \rightarrow B \rightarrow \delta C \]
is exact and \( A \cap \delta B \) is the direct sum of a divisible group and a cotorsion group.

**Proof.** Suppose the latter condition holds for an exact sequence
\[ A \rightarrow B \rightarrow C \]. Then the commutative exact diagram
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{Q},\mathbb{Q}) & \rightarrow & \text{Hom}(\mathbb{Q},\mathbb{Q}) \\
\text{Hom}(\mathbb{Q},\mathbb{Q}) & \rightarrow & \text{Hom}(\mathbb{Q},\mathbb{Q})
\end{array}
\]
implies \( \mathbb{T} \) is in \( \mathbb{T} \). Now assume \( C \) is in \( \mathbb{T} \) and \( A \rightarrow B \rightarrow C \) is in \( \mathbb{E}(C) \). The sequence \( A \cap \delta B \rightarrow B \rightarrow \delta C \) is automatically exact, and since every map from \( C \) to \( \delta C \) factors through \( A \cap \delta B \) the second map must be an epimorphism, i.e., the sequence \( A \cap \delta B \rightarrow B \rightarrow \delta C \) is exact. Now the exact commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(\mathbb{Q},\mathbb{Q}) & \rightarrow & \text{Hom}(\mathbb{Q},\mathbb{Q}) \\
\text{Hom}(\mathbb{Q},\mathbb{Q}) & \rightarrow & \text{Hom}(\mathbb{Q},\mathbb{Q})
\end{array}
\]
implies \( \text{Ext}(\mathbb{Q},A) \cong 0 \), so \( A \cap \delta B \) is a direct sum of a cotorsion group with a divisible group.

**Corollary 3.** Let \( \mathbb{T} = (\mathbb{Z},\mathbb{Z}) \). Then \( \mathbb{T} = \mathbb{C} \), \( \mathbb{T} \) has enough projectives, and a sequence \( A \rightarrow B \rightarrow C \) is in \( \mathcal{E}(C) \) if and only if the sequence of subgroups \( A \cap \delta B \rightarrow B \rightarrow \delta C \) is exact and \( A \cap \delta B \) is a direct sum of a cotorsion group with a divisible group.

\* By the rank of a group is meant the sum of the torsion free and p-ranks.
PROOF. This follows directly from Theorem 1 and 5.

It should be pointed out that the condition that the sequence
\[ A \cap B \rightarrow D_1 \rightarrow D_0 \rightarrow 0 \]
is exact is equivalent to the condition that
\[ \text{Im}(B/A) = (D_1/A) \text{ in } G. \]
Also, in the following theorems, the condition that
\[ A \rightarrow B \rightarrow C \text{ is } \pi \text{-exact } \]
be exact is equivalent to the two conditions
\[ \text{Im}(B/A) = (D_1/A) \text{ and } A \cap B = DA. \]

THEOREM 6. The group \( Z(p^n) \) is in \( \mathcal{T} \) if and only if for each sequence
\[ A \rightarrow B \rightarrow C \in \mathcal{T}(C) \]
the sequence of subgroups \( D_p \rightarrow D_0 \rightarrow 0 \) is exact.

PROOF. If the sequence of subgroups is exact it is splitting exact, and since
\[ \text{Hom}(Z(p^n), C) = \text{Hom}(Z(p^n), C_x) \]
it is clear that \( Z(p^n) \) is in \( \mathcal{T} \). Conversely, assume \( Z(p^n) \) is in \( \mathcal{T} \) and \( A \rightarrow B \rightarrow C \) is in \( \mathcal{T}(C) \). Then clearly
\[ D_1/A = (D_0/A) \text{ in } A \cap D_0 \rightarrow D_1 \rightarrow 0 \text{ is exact}. \]
The exact commutative diagram
\[ \text{Hom}(Z(p^n), A) \rightarrow \text{Hom}(Z(p^n), C) \rightarrow \text{Ext}(Z(p^n), A \cap D_0) \rightarrow 0 \]
implies that \( \text{Ext}(Z(p^n), A \cap D_0) = 0 \), which in turn implies, since \( A \cap D_0 \) is a \( \pi \)-group, that \( A \cap D_0 \) is divisible.

For future reference we point out that \( Z(p^n) \) being in \( \mathcal{T} \) implies that
\( A \cap D_0 \) is \( \pi \)-divisible. This can easily be seen by replacing \( D_0 \) by \( D_1 \) in the diagram in the above proof.

COROLLARY 4. Let \( C = (Z(p^n)) \), then \( \mathcal{T} = C_z \), \( \mathcal{T} \) has enough projectives,
and a sequence \( A \rightarrow B \rightarrow C \) is in \( \mathcal{T}(C) \) if and only if the sequence of subgroups
\[ A \cap D_0 \rightarrow D_0 \rightarrow 0 \] is exact.

PROOF. This follows directly from Theorems 1 and 6.

Finally, we characterise the proper exact sequences when all of the divisible groups are projective.

COROLLARY 5. Let \( C = (Z(p^n)) \) and \( p \) is a prime, then \( \mathcal{T} = C_z \), there
are enough projectives, and an exact sequence \( A \rightarrow B \rightarrow C \) belongs to \( \mathcal{T}(C) \)
if and only if the sequence of subgroups \( A \rightarrow D_0 \rightarrow 0 \) is exact.

PROOF. If the sequence of divisible subgroups is exact for all sequences in \( \mathcal{T}(C) \), it is clear that all divisible groups belong to \( \mathcal{T} \). Conversely, \( 0 \) being projective implies \( \text{Im}(B/A) = (D_1/A) \text{ in } G \) and \( Z(p^n) \) being projective implies \( A \cap D_0 \) is \( \pi \)-divisible for every prime \( p \) (see the remark after Theorem 6) and thus \( A \cap D_0 = DA \). This implies \( A \rightarrow D_0 \rightarrow 0 \) is exact.
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