Quotient Categories of Modules* **

By

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Gabriel and Popescu [4] have proved the following:

Theorem (*). Let \( \mathcal{C} \) be an Abelian category with exact direct limits. Let \( U \in \mathcal{C} \), \( E := \text{Hom}\_\mathcal{C}(U, U) \), \( \mathcal{A} \) be the category of right \( E \) modules, and \( S : \mathcal{C} \to \mathcal{A} \) : \( M \to \text{Hom}\_\mathcal{C}(U, M) \). Let \( T : \mathcal{A} \to \mathcal{C} \) be an adjoint of \( S \) (one exists) and \( \Phi \) the associated natural transformation from \( T \circ S \) to the identity functor on \( \mathcal{C} \). The following are equivalent:

i. \( U \) is a generator of \( \mathcal{C} \).
ii. \( S \) is completely faithful.
iii. \( \Phi \) is an isomorphism and \( T \) is exact.
iv. \( T \) is exact and induces an equivalence of \( \mathcal{A}/\text{Ker}(T) \) and \( \mathcal{C} \).

In [3] it was shown that if \( R \) is a ring and \( \mathcal{F} \) is a localizing subcategory of \( \mathcal{A} \), then \( \mathcal{A}/\mathcal{F} \) is Abelian with a generator and exact direct limits. The point of Theorem (*) is that the converse holds. Namely, if \( \mathcal{C} \) is Abelian with exact direct limits and a generator, then \( \mathcal{C} \) is equivalent to the category of all modules over a ring, modulo a localizing subcategory. Gabriel and Popescu's theorem is non-trivial even for the case \( \mathcal{C} = \mathcal{A}/R \) for some ring \( R \).

Our purpose here is to examine this situation, or more generally, the situation when \( \mathcal{C} \) is a full exact Abelian subcategory of \( \mathcal{A} \) having a generator and exact direct limits.

Let \( \mathcal{C} \) be such a category. In the notation of Theorem (*), \( T = (\cdot \otimes \mathcal{E} U) \), and Section 1 is concerned with identifying \( \text{Ker} T \). Now \( \text{Ker} T \) is a localizing subcategory of \( \mathcal{A} \) [3] (i.e., \( \text{Ker} T \) is closed under submodules homomorphic images, extensions, and arbitrary direct sums), and there is a natural one-one correspondence between localizing subcategories of \( \mathcal{A} \) and non-empty sets \( \mathcal{J} \) of right ideals of \( E \) satisfying

a) \( I \in \mathcal{J} \), \( J \) a right ideal of \( E \), \( J \supset I \) imply \( J \in \mathcal{J} \).

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a) \( I, J \in \mathcal{F} \) imply \( I \cap J \in \mathcal{F} \).

b) \( I \in \mathcal{F}, x \in E \), imply \( I : x = \{ e \in E | xe \in I \} \in \mathcal{F} \).

c) \( J \) a right ideal of \( E \), \( I \in \mathcal{F}, J : i \in \mathcal{F} \) for all \( i \in I \) imply \( J \in \mathcal{F} \).

(See [3] or [7].)

Such an \( \mathcal{F} \) is called an idempotent topologizing filter.

If \( \mathcal{F} \) is a localizing subcategory of \( \mathcal{A}_E \), the associated \( \mathcal{F} \) is just \( \mathcal{F}(\mathcal{F}) = \{ \langle I | E | I \in \mathcal{F} \} \). The point is that localizing subcategories are determined by the cyclic modules they contain. a)–d) above specify which sets of cyclics can be the cyclics of a localizing subcategory. The sets \( \mathcal{F} \) satisfying a)–d) can be quite complicated, but we show in Theorem 1.7 that if \( \mathcal{C} = \mathcal{A}_R \) then \( \mathcal{F}(\ker T) \) is just those right ideals of \( E \) containing the trace ideal \( t \) of \( U \in E \). (\( t \) is the image of the trace map \( T : U \otimes E \Hom(U, E) \to E; \otimes E f \mapsto f(t) \).

It follows that \( \ker T = \{ M \in \mathcal{A}_E | M(t) = 0 \} \). Several other properties of \( T \) are derived in Section 1. One amusing result of these is the following. If \( D \) is a division ring and \( V \) is a right vector space over \( D \), then \( \mathcal{A}_D \) is equivalent to \( \mathcal{A}_E (E = \Hom_D(V, V)) \) modulo the localizing subcategory of those \( M \in \mathcal{A}_E \) annihilated by the minimum two-sided ideal \( \{ f \in E | \dim f(V) < \infty \} \).

Section 2 is concerned with the extent to which the ring \( E = \Hom_R(U, U) \) determines \( \mathcal{C} \), \( U \) a generator in \( \mathcal{C} \). Suppose \( U \) has the property that an inclusion map \( A \rightarrow U \) induces an isomorphism \( \Hom_{\mathcal{C}}(U, U) \to \Hom_{\mathcal{C}}(A, A) \) if and only if \( A \rightarrow U \). Call such a generator a proper generator.

Every Abelian category \( \mathcal{C} \) with exact direct limits and a generator has a proper generator. In fact, if \( U \) is a generator of \( \mathcal{C} \), then the injective envelope \( \Phi \) of the direct sum of all quotients of \( U \) is a co-generator, and so \( U \otimes \Phi \) is a proper generator of \( \mathcal{C} \). One of the principal results in Section 2 is the following. If \( V \) is a proper generator in an Abelian category \( \mathcal{C}' \) with exact direct limits, and if \( E \approx E' = \Hom_{\mathcal{C}}(V, V) \), then there exists an equivalence \( F : \mathcal{C} \rightarrow \mathcal{C}' \) such that \( F(U) = V \). Let \( \mathcal{E} \) be the class of rings that are endomorphism rings of proper generators in Abelian categories with exact direct limits. The result above partitions \( \mathcal{E} \) : two rings in \( \mathcal{E} \) are in the same member of the partition if and only if they are endomorphism rings of proper generators in equivalent Abelian categories with exact direct limits. This should have some ring theoretical significance. It is shown in Corollary 2.3 that a ring \( R \) is in \( \mathcal{E} \) if and only if the set of right ideals \( I \) such that the natural map \( \Hom_R(R, R) \to \Hom_{\mathcal{C}}(I, R) \) is an isomorphism satisfies properties a)–d) above for idempotent topologizing filters. The class \( \mathcal{C} \) contains all commutative rings (Theorem 2.5) and all self-injective rings. Several other properties of \( \mathcal{E} \) are exhibited in Section 2.
Using the notation in Theorem (1) above it is shown in Section 3 that \( E \) is right self-injective if and only if \( U \) is injective. Also, the injective envelope of \( U \) is \( T(\mathcal{E}) \), where \( \mathcal{E} \) is the injective envelope of \( E \) in \( \mathcal{E} \).

In particular, if \( U \) is a generator in the category \( \mathcal{E} \) of right \( R \)-modules, the injective envelope of \( U \) is \( E \otimes R U \).

The category of Abelian \( p \)-groups is Abelian with a generator and exact direct limits. This category is the topic of Section 4. An Abelian \( p \)-group \( G \) such that \( G \) modulo its maximum divisible subgroup is unbounded turns out to be flat as a module over its endomorphism ring (Theorem 4.1). This fact is of some group theoretical interest [5]. It seems to be a bit difficult to determine which groups are proper generators in the category of Abelian \( p \)-groups.

1. The Kernel of \( T \)

Throughout, \( \mathcal{F} \) will be an Abelian category with a generator \( U \) and exact direct limits, and \( E \) will be the ring \( \text{Hom}_R(U, U) \). The category \( \mathcal{F} \) has infinite sums and injective envelopes, and the functor

\[ S = \text{Hom}_R(U, \cdot) : \mathcal{F} \to \mathcal{E} \]

has an adjoint [3] which will be denoted by \( T \).

**Definition.** Let \( I \) be a right ideal in \( E \). Then \( IU \) is the image of the map

\[ \sum_{i \in I} U \rightarrow U. \]

As pointed out in the introduction, to identify \( \text{Ker} T \) is the same as identifying \( f \), the set of right ideals \( I \) of \( E \) such that \( T(E)I = 0 \). The following proposition does this.

**Proposition 1.1.** A right ideal \( I \) of \( E \) is in \( \mathcal{F} \) (i.e., \( T(E)I = 0 \)) if and only if \( IU = U \).

**Proof.** Since \( T \) is right exact, \( T(E)I = 0 \) if and only if the inclusion \( I \to E \) induces an epimorphism \( T(I) \to T(E) \). But the image of this map is just \( IU \). To see this, let \( i \in I \). Then \( S(i) : E \to E : e \to ie \). Since \( S(i)(E) \subseteq I \), \( TS(i) : T(E) \to T(E) \) can be factored through \( T(I) \). In fact \( \text{Im} \left( \sum_{i \in I} T(E) \rightarrow T(E) \right) = T(I) \), since \( T \) is exact and commutes with direct sums. In the commutative diagram

\[
\begin{array}{ccc}
\sum_{i \in I} T(E) & \xrightarrow{\pi_i} & T(E) \\
\downarrow \bigwedge & & \downarrow \bigwedge \\
\sum_{i \in I} U & \xrightarrow{\sum_{i \in I} \pi_i} & U
\end{array}
\]

where

\[ \bigwedge \]
the vertical arrows are isomorphisms, by Theorem (1) of Gabriel and Popescu. It follows that the image of the composition \( T(I) \to T(E) \xrightarrow{\sim} U \) is \( I \).

**Proposition 1.2.** If \( I \in \mathfrak{F} \), then \( \text{Hom}_R(E,I,E) = 0 \) and \( \text{Ext}_R^1(E,I,E) = 0 \).

**Proof.** \( \text{Hom}_R(E,I,E) = \text{Hom}_R(E,I,S(U)) \approx \text{Hom}_R(T(E/I),U) = 0 \) since \( T(E/I) = 0 \). In the commutative diagram with exact row

\[
\begin{array}{cccc}
\text{Hom}_R(E,E_j) & \to & \text{Hom}_R(E_I, E) & \to & \text{Ext}_R^1(E/I,E) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Hom}_R(T(E), U) & \to & \text{Hom}_R(T(I), U) & & & & 
\end{array}
\]

the vertical maps and the lower horizontal map are isomorphisms. It follows that \( \text{Ext}_R^1(E/I,E) = 0 \).

**Corollary 1.3.** If \( I \in \mathfrak{F} \), then the left annihilator of \( I \) is zero and \( I \) is essential in \( E \) as a right ideal.

**Proof.** The left annihilator of \( I \) is isomorphic to \( \text{Hom}_R(E,I,E) \), which is zero by Proposition 1.2 above. Let \( J \) be a right ideal of \( E \) and suppose \( I \in \mathfrak{F} \) such that \( I \cap J = 0 \). Then \( I \oplus J \in \mathfrak{F} \), since \( \mathfrak{F} \) is a filter, and by Proposition 1.2 above, the sequence

\[
0 = \text{Hom}_R(E/I,E) \to \text{Hom}_R((I \oplus J)/I,E) \to \text{Ext}_R^1(E/(I \oplus J),E) = 0
\]

is exact, implying \( \text{Hom}_R(I \oplus J/I,E) = 0 \). It follows immediately that \( J = 0 \).

If \( U \) is a generator in \( \mathfrak{S}_R \) and \( A \) is a right \( R \)-module then

\[
\text{Ext}_R^1(U,A) \otimes_R U = 0.
\]

A more general statement is the following.

**Proposition 1.4.** If \( U \) is a generator in \( \mathfrak{C} \), \( A \in \mathfrak{C} \) then \( \text{Ext}_R^1(U,A) \in \text{Ker} T \).

**Proof.** Let \( 0 \to A \to Q \to C \to 0 \) be exact in \( \mathfrak{C} \) with \( Q \) projective. This yields an exact sequence

\[
0 \to \text{Hom}_R(U,A) \to \text{Hom}_R(U,Q) \to \text{Hom}_R(U,C) \to \text{Ext}_R^1(U,A) \to 0
\]

of right \( E \)-modules. Since \( T: \mathfrak{S}_R \to \mathfrak{C} \) is exact, and \( T \circ S \approx 1_\mathfrak{C} \), this leads to an exact commutative diagram

\[
\begin{array}{cccc}
0 & \to & T \circ S(A) & \to & T \circ S(Q) & \to & T \circ S(C) & \to & T \circ \text{Ext}_R^1(U,A) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & Q & \to & C & \to & 0
\end{array}
\]

with the vertical arrows isomorphisms. In particular, \( T(\text{Ext}_R^1(U,A)) = 0 \).
Definition. Let $R$ be a ring, $U \in \mathcal{M}_R$. The trace ideal of $U$ is the ring $E = \text{Hom}_R(U, U)$ is the image of the trace map

$$U \otimes_R \text{Hom}_R(U, E) \to E: u \otimes f \mapsto f(u).$$

The trace map is a two-sided $E$-map, so the trace ideal is a two-sided ideal of $E$. Note that the trace ideal of $U$ in $E$ is also the image of the map

$$\sum_{f \in \text{Hom}_R(U, E)} U \otimes f E.$$

If $\mathcal{C}$ is a full exact subcategory of $\mathcal{M}_R$, then the trace ideal of the generator $U$ of $\mathcal{C}$ is defined. In this case, the trace ideal of $U$ in $E$ will be denoted by $I$.

Proposition 1.5. Let $\mathcal{C}$ be a full, exact subcategory of $\mathcal{M}_R$ for some ring $R$. Then $I \subseteq \bigcap_{I \in \mathcal{C}} I$.

Proof. Let $f \in \text{Hom}_R(U, E)$ and $I \in \mathcal{C}$. Then

$$f(U) = f(I U) = f f(U) \subseteq I.$$

Let $\Gamma = \bigcap_{I \in \mathcal{C}} I$. In the case $\mathcal{C} = \mathcal{M}_R$, it is shown below that $t \in \mathcal{C}$, so that $t = \Gamma$. In Section 4 it is shown that $t \rightarrow \Gamma$ in the case $\mathcal{C}$ is the category of all Abelian $p$-groups for some prime $p$. It is not known whether it is always the case that $t = \Gamma$. If this should be true, it would permit a generalization of the trace ideal to the case of a generator in an Abelian category with exact direct limits. The following proposition gives a faint hint that these two ideals might be the same.

Proposition 1.6. The ideal $\Gamma = \bigcap_{I \in \mathcal{C}} I$ is a two-sided ideal of $E$.

Proof. Let $x \in \Gamma$, $e \in E$, $I \in \mathcal{C}$. Then $I : e \in \mathcal{C}$ implies $x \in I : e$, so that $x e \in I$.

Theorem 1.7. If $U$ is a generator in $\mathcal{M}_R$, then the trace ideal $t$ of $U$ in $E$ has the following properties:

i. $t U = U$. (Thus $E/I \in \text{Ker } T$ if and only if $I \supseteq t$.)

ii. $t^2 = t$.

iii. The left annihilator of $t$ is 0.

iv. $t$ is finitely generated as a two-sided ideal.

v. $t$ is essential as a right ideal.

Proof. By the Dual Basis Lemma, there exist maps

$$f_1, \ldots, f_n \in \text{Hom}_R(U, E)$$
and elements $u_1, \ldots, u_n \in U$ such that for any $u \in U$,

$$u = \sum_{i=1}^{n} f_i(u)(u_i).$$

But $f_i \subseteq U$ for each $i$, so $u \subseteq t(U)$ for any $u \in U$. Thus $tU = U$. The remainder of $i.$ follows from Propositions 1.1 and 1.5. Now ii. and iv. follow from Corollary 1.3, since $t \in f$. Parts ii. and iv. appear in [I]. However, $\beta = 1$ follows immediately from the fact that $\beta \in f$ and $t \subseteq \bigcap_{i \neq j} I_i$.

To restate some of the results above, if $U$ is a generator in $\mathcal{M}_R$ and $t$ is the trace ideal of $U$ in $E$, then for $M \in \mathcal{M}_R$, $M \otimes_R U = 0$ if and only if $Mt = 0$, and $\mathcal{M}_R$ is isomorphic to the category $\mathcal{M}_R$ modulo the class of modules $M$ such that $Mt = 0$.

It is interesting to note that $U$ is a generator in $\mathcal{M}_R$ if and only if $T$ is an equivalence $\mathcal{M}_R \to \mathcal{M}_R$.

Some of the members of $f$ can be described in terms of families of sub-objects of $U$, and in the special case $G = \mathcal{M}_R$ and $U$ is projective the trace ideal can be described in terms of the set of finitely generated sub-modules of $U$. This is shown in the next two propositions.

**Proposition 1.8.** Let $\gamma$ be a set of subobjects of $U$ which generate $U$ and let $I_\gamma$ be the right ideal of $E$ generated by $\{f \in E | f(U) \subseteq S$ for some $S \in \gamma\}$. Then $I_\gamma \subseteq \varepsilon$.

**Proof.** Since $U$ is a generator,

$$\sum_{f \in \gamma} U^{S_f} = S$$

is an epimorphism for each $S$, and by the hypothesis, the inclusion maps $S \to U$ induce an epimorphism

$$\sum_{S \in \gamma} S \to U.$$

The composition of these maps

$$\sum_{S \in \gamma} \left( \sum_{f \in \gamma} U^{S_f} \right) \to S \to U$$

is an epimorphism with image $I_\gamma U$. Thus $I_\gamma \subseteq \varepsilon$.

This proposition, when $U$ is an $R$-module, says the set of all endomorphisms which carry $U$ into cyclic submodules of $U$ generate a member of $\mathcal{J}$. Also the set of endomorphisms which carry $U$ into finitely generated sub-modules of $U$ generate a member of $\mathcal{J}$.

**Proposition 1.9.** Suppose $U$ is a projective generator in $\mathcal{M}_R$. Let $A$ be
the right ideal of $E$ generated by \{ $f \in E$ | $f(U) \subseteq U$ \} and $\Omega = \{ f \in E | f(U) \subseteq \text{fin gen} \text{mod} U \}$. Then $t = A = \Omega$.

**Proof.** From previous propositions, $t \in A$. Clearly $A \subseteq \Omega$. Suppose $f \in E$ with $f(U) \subseteq uR$ for some $u \in U$. Since $U$ is projective and $R \to uR$ is an epimorphism, there exists a homomorphism $g : U \to R$ such that $ug(x) = f(x)$ for all $x \in U$. Define $\sigma : U \to E$ by $\sigma(u)(x) = g(x)$ for $x, y \in U$. Then for $e \in E$, $\sigma(u)(y)(x) = (\sigma(y)(x)) e = (\sigma(y)(x)) e = e(\sigma(y)(x)) = e(\sigma(y)(x))$. It follows easily that $\sigma$ is a left $E$ homomorphism. Then $f = \sigma(u)$, implying $f \in t$. Thus $A \subseteq \Omega$, so $A = t$.

Let $f \in \Omega$, $f(U) \subseteq u_1 R \oplus \cdots \oplus u_n R$. Since $U$ is projective, $f$ can be lifted to a map $g : U \to u_1 R \oplus \cdots \oplus u_n R$. Let $e_i$ be the composition of $g$ with the $i$-th projection. Then $f = e_1 + \cdots + e_n \in A$, since $e_i \in A$ for each $i$.

**Corollary 1.10.** Let $R$ be a division ring, $U \in \mathcal{A}_R$, $E = \text{Hom}_R(U, U)$. Let $I$ be the minimum two-sided ideal $I \subseteq E \subseteq \text{N}_E$, and let $\mathcal{F}$ be the subcategory $\{ M \in \mathcal{M}_E | MI = 0 \}$. Then there exists an equivalence $\mathcal{A}_R \sim \mathcal{M}_E / \mathcal{F}$.

**Proof.** By Proposition 1.9 above, $I = t$, and an application of Theorem 1.7 concludes the proof.

2. Endomorphism rings of proper generators

A ring can be the endomorphism ring of generators $U$ and $U'$ of Abelian categories $\mathcal{C}$ and $\mathcal{C}'$ with exact direct limits without $\mathcal{C}$ and $\mathcal{C}'$ being equivalent. For example, if $R$ is a division ring and $U$ is an infinite dimensional vector space over $\mathcal{A}_R$, then its endomorphism ring $E$ is also the endomorphism ring of the generator $E \in \mathcal{A}_R$. However, $\mathcal{A}_R$ and $\mathcal{A}_E$ are not equivalent. The point is that $E$ is not a proper generator in $\mathcal{A}_R$. The main objective here is to show that if $U$ and $U'$ are proper generators of Abelian categories $\mathcal{C}$ and $\mathcal{C}'$ with exact direct limits, and if $U$ and $U'$ have isomorphic endomorphism rings then $\mathcal{C}$ and $\mathcal{C}'$ are equivalent.

**Lemma 2.1.** Let $R$ be a ring and let $\mathcal{F}$ be the set of right ideals $I$ of $R$ such that the inclusion $I \to R$ induces an isomorphism $\text{Hom}_R(R, R) \to \text{Hom}_R(I, R)$. If $\mathcal{F}$ is an idempotent topologizing filter then $R$ is a proper generator in the quotient category $\mathcal{D} = \mathcal{A}_R / \mathcal{F}$.

**Proof.** Let $f : M \to R$ be an inclusion map in $\mathcal{D}$ which induces an isomorphism $\text{Hom}_R(R, R) \to \text{Hom}_R(M, R)$. Let $f' : M' \to R/S$ be a representative of this map in $\mathcal{A}_R$, where $M/M' \in \mathcal{F}(\mathcal{F})$ and $S \in \mathcal{F}(\mathcal{F})$. Since $\text{Hom}_R(R/I, R) = 0$ for all $I \in \mathcal{F}$, $R$ has no non-zero submodules in $\mathcal{F}(\mathcal{F})$, so $S = 0$. Let $\mathcal{F} = \mathcal{F}(\mathcal{F})$. Then $M \cong R$ in $\mathcal{D}$ if and only if
Consider the exact commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_R(R/J, R) \rightarrow \text{Hom}_R(R, R) \rightarrow \text{Hom}_R(J, R) \rightarrow \text{Ext}^1_R(R/J, R) \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_{\mathcal{E}}(R, R) \rightarrow \text{Hom}_{\mathcal{E}}(J, R) & & 0.
\end{array}
\]

The map \( \text{Hom}_R(R, R) \rightarrow \text{Hom}_{\mathcal{E}}(R, R) \) is an isomorphism, from the definition of \( \mathcal{E} \), so that \( \text{Hom}_R(R/J, R) = 0 \). Since the diagram commutes, the other vertical arrow is an isomorphism so \( \text{Ext}^1_R(R/J, R) = 0 \) and \( J \in \mathcal{E} \).

**Proposition 2.2.** Let \( \mathcal{F}(E) \) be the set of right ideals of \( E \) such that \( \text{Hom}_R(E/I, E) = 0 \) and \( \text{Ext}^1_R(E/I, E) = 0 \). Then \( U \) is a proper generator in \( \mathcal{E} \) if and only if \( \mathcal{F}(E) = \mathcal{F} \).

**Proof.** If \( \mathcal{F}(E) = \mathcal{F} \), Lemma 2.1 above shows that \( U \) is a proper generator in \( \mathcal{E} \) (using the equivalence \( \mathcal{E} \sim \mathcal{G} \) of Gabriel-Popescu's Theorem (1)).

Suppose \( U \) is a proper generator in \( \mathcal{E} \). If \( I \) is a right ideal of \( E \), the sequences

\[
0 \rightarrow T(I) \rightarrow T(E) \rightarrow T(E/I) \rightarrow 0
\]

and

\[
0 \rightarrow I U \rightarrow U \rightarrow U/I \rightarrow 0
\]

are equivalent, by Proposition 1.1. Thus the adjoint maps lead to a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_E(U/I, U) \rightarrow \text{Hom}_E(U, U) \rightarrow \text{Hom}_E(U/I, U) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_E(E/I, E) \rightarrow \text{Hom}_E(E, E) \rightarrow \text{Ext}^1_R(E/I, E) & & 0
\end{array}
\]

with the vertical maps isomorphisms and the rows exact. Hence \( I \in \mathcal{F}(E) \) if and only if \( \text{Hom}_E(U/I, U) \rightarrow \text{Hom}_E(U/I, U) \) is an isomorphism, and this map is an isomorphism if and only if \( I \in \mathcal{F} \) (since \( U \) is proper).

From the two previous propositions one has immediately

**Corollary 2.3.** A ring \( R \) is the endomorphism ring of a proper generator of an Abelian category with exact direct limits if and only if the set of right ideals \( I \) of \( R \) for which the map \( \text{Hom}_R(R, R) \rightarrow \text{Hom}_R(I, R) \) is an isomorphism is an idempotent topologizing filter.

If \( R \) is self-injective it is easy to see that \( \mathcal{F}(R) \) is an idempotent topologizing filter. In this case, \( \mathcal{F}(R) \) is the set of right ideals \( I \) such that \( \text{Hom}_R(R/I, R) = 0 \). The next proposition allows us to describe other classes of rings which are included in \( \mathcal{F} \), the class of all rings which are endomorphism rings of proper generators.
Proposition 2.4. Let \( R \) be a ring. If \( I \in \mathcal{J}(R) \) implies \( I \cdot x \in \mathcal{J}(R) \) for all \( x \in R \), then \( \mathcal{J}(R) \) is an ideal-preserving ring.

Proof. If \( I \in \mathcal{J}(R) \) and \( J \supseteq I \) a right ideal of \( R \), the exact sequence \( E/I \rightarrow R/J \rightarrow 0 \) induces an exact sequence \( 0 \rightarrow \text{Hom}_R(R/J, R) \rightarrow \text{Hom}_R(R/I, R) \rightarrow 0 \). Thus \( \text{Hom}_R(R/J, R) = 0 \).

Let \( I \in \mathcal{J} \) and suppose \( J \) is a right ideal of \( R \) such that \( J \cdot x \in \mathcal{J} \) for all \( x \in I \). Since \( R/J \cdot x \approx (xR + J)/J \), \( \text{Hom}_R((xR + J)/J, R) = 0 \) for all \( x \in I \). Thus \( \text{Hom}_R(I + J)/J, R) = 0 \). The exact sequence \( 0 \rightarrow (I + J)/J \rightarrow R/J \rightarrow R/(I + J) \rightarrow 0 \) induces an exact sequence \( \text{Hom}_R(R/(I + J), R) \rightarrow \text{Hom}_R(R/J, R) \rightarrow \text{Hom}_R(I + J)/J, R) = 0 \).

But the first term is also zero by the argument of the preceding paragraph, so \( \text{Hom}_R(R/J, R) = 0 \).

Let \( I \in \mathcal{J} \) and \( J \supseteq I \). The exact sequence \( 0 \rightarrow J/I \rightarrow R/I \rightarrow R/J \rightarrow 0 \) induces the exact sequence \( 0 \rightarrow \text{Hom}_R(R/I, R) \rightarrow \text{Hom}_R(R/J, R) \rightarrow \text{Ext}_R^1(R/J, R) \rightarrow 0 \).

and since for \( x \in J, I : x \in \mathcal{J} \) and \( R/(I : x) \approx (xR + I)/I \), \( \text{Hom}_R((xR + I)/I, R) = 0 \) for all \( x \in J \). It follows that \( 0 = \text{Hom}_R(J/I, R) \) and hence \( 0 = \text{Ext}_R^1(R/J, R) \).

Let \( I, J \in \mathcal{J} \). The exact sequence \( 0 \rightarrow (I + J)/J \rightarrow R/(I \cap J) \rightarrow R/I \rightarrow 0 \) yields an exact sequence \( 0 = \text{Hom}_R(R/I, R) \rightarrow \text{Hom}_R((I + J)/J, R) \rightarrow \text{Hom}_R((I + J)/J, R) \rightarrow \text{Ext}_R^1(R/J, R) \rightarrow 0 \).

and the exact sequence \( 0 \rightarrow (I + J)/J \rightarrow R/J \rightarrow R/(I + J) \rightarrow 0 \) yields an exact sequence \( 0 = \text{Hom}_R(R/J, R) \rightarrow \text{Hom}_R((I + J)/J, R) \rightarrow \text{Ext}_R^1(R/J, R) \rightarrow 0 \).

By the previous paragraph, \( \text{Ext}_R^1(R/I, R) = 0 \), so that \( \text{Hom}_R(R/(I \cap J), R) = 0 \).

The exact sequence \( 0 \rightarrow (I + J)/J \rightarrow R/(I \cap J) \rightarrow R/I \rightarrow 0 \) yields an exact sequence \( 0 = \text{Ext}_R^1(R/I, R) \rightarrow \text{Ext}_R^1(R/(I \cap J), R) \rightarrow \text{Ext}_R^1((I + J)/J, R) \).
Now for 
\( x \in R, J : x \in \mathcal{J} \) so \( \text{Ext}_k^1(R/(J:x), R) = 0 \), so \( \text{Ext}_k^1((x R + J)/(J, R)) = 0 \).

Now consider the exact sequence

\[
0 \rightarrow K \rightarrow \sum_{x \in \mathcal{J}} (x R + J)/(J) \rightarrow (I + J)/(J) \rightarrow 0
\]

where the maps \((x R + J)/(J)/(I + J)/(J)\) are the inclusion maps. This yields an exact sequence

\[
0 = \text{Hom}_R(K, R) \rightarrow \text{Ext}_k^1((I + J)/(J), R) \rightarrow \text{Ext}_k^1\left( \sum_{x \in \mathcal{J}} (x R + J)/(J), R \right) \approx \prod_{x \in \mathcal{J}} \text{Ext}_k^1((x R + J)/(J), R) = 0.
\]

Thus \( \text{Ext}_k^1(R/(I \cap J), R) = 0 \). Let \( I \in \mathcal{J} \) and suppose \( J : x \in \mathcal{J} \) for all \( x \in \mathcal{I} \), and some right ideal \( J \) of \( R \). As in the above paragraph,

\[
\text{Ext}_k^1((I + J)/(J), R) = 0.
\]

The exact sequence

\[
0 \rightarrow ((I + J)/(J) \rightarrow R/(J) \rightarrow R/(I + J) \rightarrow 0
\]

leads to an exact sequence

\[
\text{Ext}_k^1(R/(I + J), R) \rightarrow \text{Ext}_k^1(R/(J), R) \rightarrow 0 = \text{Ext}_k^1(R/(I + J), R).
\]

But the first term is also zero, by the third paragraph of this proof. Thus \( \text{Ext}_k^1(R/(J), R) = 0 \).

**Remark.** Suppose \( I \in \mathcal{I}, J \supset I \) imply \( \text{Ext}_k^1(R/J, R) = 0 \). Let \( J \in \mathcal{J} \), \( x \in R \). The exact sequence

\[
0 \rightarrow R/(I : x) \rightarrow R/I \rightarrow R/(x R + I) \rightarrow 0
\]

yields an exact sequence

\[
0 \rightarrow \text{Hom}_R(R/I : x, R) \rightarrow \text{Ext}_k^1(R/(x R + I), R) = 0.
\]

It follows that the hypothesis

"\( I \in \mathcal{I}, J \supset I, x \in R \) imply \( \text{Ext}_k^1(R/J, R) = 0 \) and \( \text{Ext}_k^1(R/I : x, R) = 0" \]

would also imply that \( \mathcal{J}(R) \) is an idempotent topologizing filter.

It is interesting to note that if \( \mathcal{J}(R) \) is a filter then every \( I \in \mathcal{J}(R) \) is an essential right ideal of \( R \). To see this, let \( I \in \mathcal{J}(R) \) and suppose \( J \) is a right ideal of \( R \) with \( I \cap J = 0 \). The exact sequence \( 0 \rightarrow J \rightarrow R/I \rightarrow R/(I \oplus J) \rightarrow 0 \) yields an exact sequence

\[
0 = \text{Hom}_R(R/I, R) \rightarrow \text{Hom}_R(R, R) \rightarrow \text{Ext}_k^1(R/(I \oplus J), R).
\]

If \( \mathcal{J}(R) \) is a filter then \( I \oplus J \in \mathcal{J}(R) \), implying \( \text{Ext}_k^1(R/(I \oplus J), R) = 0 \).
It follows that \( \text{Hom}_R(I, R) = 0 \), which can happen only if \( J = 0 \).

**Theorem 2.5.** If \( R \) is a commutative ring then \( R \) is the endomorphism ring of a proper generator.

**Proof.** Let \( I \in \mathcal{F}(R) \), \( x \in R \). Since \( I : x \supseteq I \), \( \text{Hom}(R(I : x), R) = 0 \).

For \( y \in I : x/I \), \((yR + I)/I \cong R/I(y) \), so \( \text{Hom}_R((yR + I)/I, R) = 0 \).

It follows that \( \text{Hom}_R(I : x)/I, R) = 0 \). Now the exact sequence

\[
0 \to (I : x)/I \to R/I \to (R(I : x)) \to 0
\]

yields the exact sequence

\[
0 = \text{Hom}_R((I : x)/I, R) \to \text{Ext}^1_B(R/I : x, R) \to \text{Ext}^1_B(R/I, R) = 0.
\]

Thus \( \text{Ext}^1_B(R/I : x, R) = 0 \).

**Proposition 2.6.** If \( R \) is a principal ideal domain then \( R \) is a proper generator in \( \mathcal{F}_R \).

**Proof.** Let \( I \) be an ideal of \( R \) and suppose \( 0 \neq I \neq R \). Let \( Q \) be the quotient field of \( R \). Since \( R \) is a principal ideal domain and \( I = R \), \( I \subseteq xR \)
with \( x^{-1} \notin R \). Now \( j : I \to R : x \mapsto r \) is an \( R \)-homomorphism which cannot be extended to an endomorphism of \( R \), so \( \text{Hom}_R(R, R) \to \text{Hom}_R(I, R) \) is not an isomorphism. It follows that \( \mathcal{F}(R) = \{ R \} \), and \( R \) is a proper generator in \( \mathcal{F}_R \).

**Example.** Let \( F \) be a field, \( R = F[x, y] \) and let \( I \) be the ideal in \( R \)
generated by \( x \) and \( y \). Any homomorphism \( I \to R \) is a multiplication by an element \( p/q \) of the quotient field, where \( p \in I \) and \( q \neq 0 \). Then \( x \) divides both \( p \) and \( q \). Suppose \( x \) does not divide \( p \). Then \( x \) does not divide \( q \). If \( x \) does divide \( q \), then \( x \) divides both \( p \) and \( q \). Thus, \( x \) does not divide \( q \).

**Proposition 2.7.** If \( R \) is a hereditary ring then \( R \) is the endomorphism ring of a proper generator if and only if \( I \in \mathcal{F}(R) \), \( J \supseteq I \) implies \( J \in \mathcal{F}(R) \).

**Proof.** Let \( I \in \mathcal{F}(R) \), \( x \in R \). The exact sequence

\[
0 \to R(I : x) \to R/I \to R(xR + I) \to 0
\]
yields an exact sequence

\[
\text{Hom}_R(R/I, R) = 0 \to \text{Hom}_R(R(I : x), R) \to \text{Ext}^1_B(R/xR + I, R) \to 0.
\]
\[
\text{Ext}^1_F(R/I, R) = 0 \rightarrow \text{Ext}^1_F(R/(x \cdot I), R) \rightarrow \text{Ext}^1_F(R/(x R + I), R) = 0.
\]
Since \(x R + I \subseteq I\), if one assumes \(\mathcal{I}\) (i) includes every right ideal of \(R\) which contains some member of \(\mathcal{I}(R)\), \(\text{Ext}^1_F(R/(x R + I), R) = 0\). Thus every term of the above sequence is zero and \(I \cdot x \in \mathcal{I}(R)\). By Proposition 24, \(\mathcal{I}(R)\) is an idempotent topologizing filter. Apply Corollary 2.3. The converse also follows from Corollary 2.3.

**Theorem 2.8.** Let \(\mathcal{E}\) and \(\mathcal{E}'\) be Abelian categories with exact direct limits, and let \(U\) and \(U'\) be proper generators of \(\mathcal{E}\) and \(\mathcal{E}'\), respectively. If the rings \(\text{Hom}_\mathcal{E}(U, U)\) and \(\text{Hom}_\mathcal{E}'(U', U')\) are isomorphic, then there exists a categorical equivalence \(F: \mathcal{E} \rightarrow \mathcal{E}'\) such that \(F(U) = U'\).

**Proof.** By Proposition 2.2, \(\mathcal{I} = \mathcal{I}(E)\), so by Gabriel and Popescu's theorem (*), \(T\) induces an equivalence \(\mathcal{A}_E(\mathcal{I}(E)) \rightarrow \mathcal{E}\). Let \(T^{-1}\) denote the inverse of this equivalence. Then \(T^{-1}(U) = E\). Similarly there is an equivalence \(T' : \mathcal{A}_E'(\mathcal{I}(E')) \rightarrow \mathcal{E}'\) with \(T'(E) = U'\). The isomorphism \(E \cong E'\) induces an equivalence \(G : \mathcal{A}_E \rightarrow \mathcal{A}_E'\) with \(G(E) = E'\). Now \(I \in \mathcal{I}(E)\) if and only if \(G(I) \in \mathcal{I}(E')\), for since \(G\) is exact, \(E' / G(I) \cong G(E)/I\), so that \(\text{Hom}_\mathcal{E}(E/I, E) \cong \text{Hom}_\mathcal{E}'(E'/G(I), E')\) and

\[
\text{Ext}^1_F(E/I, E) \cong \text{Ext}^1_F(E'/G(I), E').
\]

Thus \(G\) induces an equivalence \(G : \mathcal{A}_E(\mathcal{I}(E)) \cong \mathcal{A}_E'(\mathcal{I}(E'))\) with \(G(E) = E'\). Now the composition \(T' \circ G \circ T^{-1}\) yields the desired equivalence.

**3. Injectives and Injective Envelopes**

Here the behavior of injectives under the action of the functors \(S : \mathcal{E} \rightarrow \mathcal{A}_E\) and \(T : \mathcal{A}_E \rightarrow \mathcal{E}\) is determined. Briefly, \(S\) preserves injective envelopes (Corollary 3.3), and if \(\tilde{X}\) denotes the injective envelope of \(X\), then \(T(S(X)) = \tilde{X}\) (Proposition 3.4). It follows that \(E\) is right self-injective if and only if \(U\) is injective (Theorem 3.5).

**Proposition 3.1.** If \(J \in \mathcal{E}\) is injective, then \(S(A) = \text{Hom}_\mathcal{E}(U, A) : E\)-module is an injective right \(E\)-module.

**Proof.** Let

\[
0 \rightarrow M \xrightarrow{i} N \xrightarrow{\phi} S(A)
\]

be an exact diagram in \(\mathcal{A}_E\). This induces an exact diagram

\[
0 \rightarrow T(M) \xrightarrow{j} T(N) \xrightarrow{\phi} T(S(A)) \xrightarrow{g} A
\]

where \(g\) is the morphism induced by \(\phi\). Since \(E\) is injective, \(N\) can be extended to an exact sequence

\[
0 \rightarrow M \xrightarrow{i} N \xrightarrow{\phi} S(A) \rightarrow 0.
\]

This shows that \(T(S(A))\) is an injective \(E\)-module.
in $\mathcal{V}$, which can be completed since $A$ is injective. Let
\[\Phi(X, Y) : \text{Hom}_{\mathcal{V}}(T(X), Y) \to \text{Hom}_{\mathcal{E}}(X, S(Y))\]
denote the isomorphism associated with the adjoint functors $S$ and $T$. Then $\Phi : TS \to 1$ by $\Phi_X = \varphi(S(Y)), Y^{-1}(1_{\mathcal{S}(Y)})$ for $Y \in \mathcal{V}$. Let $\mathcal{T} : 1_{\mathcal{E}} \to ST$ be the natural transformation given by
\[\mathcal{T}_X = \varphi(X, T(X))(1_{T(X)})\]
for $X \in \mathcal{E}$. We have the following diagram in $\mathcal{E}$:

\[
\begin{array}{c}
0 \to M \xrightarrow{i} N \\
\downarrow S(y) \circ \mathcal{T}_N \\
S(A)
\end{array}
\]

It remains to show that it commutes. We will refer to the following commutative diagram.

\[
\begin{array}{c}
\text{Hom}_{\mathcal{V}}(T(S(A), A) \xrightarrow{\varphi}(S(A), A) \\
\downarrow \text{Hom}_{\mathcal{V}}(T(f), A) \\
\text{Hom}_{\mathcal{E}}(S(A), A) \\
\downarrow \text{Hom}_{\mathcal{E}}(f, A) \\
\text{Hom}_{\mathcal{E}}(S(A), A)
\end{array}
\]

Now

\[
\begin{align*}
\text{Hom}_{\mathcal{E}}(f, S(A)) \varphi(S(A), A)(\mathcal{T}_A) &= f = \varphi(M, A) \text{Hom}_{\mathcal{E}}(T(f), A)(\mathcal{T}_A) \\
&= \varphi(M, A)(\mathcal{T}_A)(f) = \varphi(M, A)(g T(i)) = \varphi(M, A) \text{Hom}_{\mathcal{E}}(T(i), A)(g) \\
&= \text{Hom}_{\mathcal{E}}(i, S(A)) \varphi(N, A)(g) = \varphi(N, A)(g) i = \\
&= \varphi(N, A)(\text{Hom}_{\mathcal{E}}(T(N), g)(1_{T(N)})) i \\
&= \text{Hom}_{\mathcal{E}}(N, S(g))(\varphi(N, T(N))(1_{T(N)})) i = S(g) \mathcal{T}_N i,
\end{align*}
\]

concluding the proof.

**Proposition 3.2.** If $A$ is an essential extension of $B$ in $\mathcal{V}$ then $S(A)$ is an essential extension of $S(B)$ in $\mathcal{E}$.

**Proof.** Let $f \in S(A) = \text{Hom}_{\mathcal{E}}(U, A)$ with $j + 0$. Let $K \xrightarrow{f} U$ be a kernel of the composition $U \xrightarrow{j} A \to A/B$ (i.e., $K = f^{-1}(B)$). Since $A$ is an essential extension of $B$, $K + 0$. Thus there is a map $g = 0, g : U \to U$
with $\text{Im} \ f \subset K$. Now $0 \neq g : U \to B$. Thus $0 \neq f \in S(A)$ implies $0 \neq f(E) \cap S(B)$, and $S(B)$ is essential in $S(A)$.

**Corollary 3.3.** The functor $S$ preserves injective envelopes.

**Proposition 3.4.** If $M \in \mathcal{M}$ is isomorphic to $S(X)$ for some $X \in \mathcal{E}$, and $\mathcal{A}$ is an injective envelope of $M$ in $\mathcal{M}$, then $T(\mathcal{A})$ is an injective envelope of $T(M)$ in $\mathcal{E}$.

**Proof.** By the previous propositions, if $A$ is an injective envelope of $T(M)$ in $\mathcal{E}$, then $S(A)$ is an injective envelope of $ST(M)$ in $\mathcal{M}$. Now since $M \cong S(X)$, $ST(M) \cong STS(X) \cong S(X) \cong M$. Thus $S(A) \cong \mathcal{A}$, and $T(\mathcal{A}) \cong T(S(A)) \cong A$ is an injective envelope of $T(M)$ in $\mathcal{E}$.

If $U$ is a generator in $\mathcal{M}$ for some ring $R$, the previous proposition says that $E \otimes_R U$ is an injective envelope of $U$ in $\mathcal{M}$.

**Theorem 3.5.** The ring $E$ is right self-injective if and only if $U$ is injective in $\mathcal{E}$.

**Proof.** If $E$ is right self-injective then $U \cong T(U) = T(E) = U$ by proposition 3.4.

Assume $U$ is injective in $\mathcal{E}$. Then by Corollary 3.3, $E \cong S(U) = S(U) \cong E$ so $E$ is right self-injective.

**Corollary 3.6.** If $U$ is a generator in $\mathcal{M}$, and if $E$ is right self-injective, then the ring $R$ is right self-injective.

**Proof.** By the theorem above, $U$ is injective. But $R$ is isomorphic to a summand of a finite sum of copies of $U$.

### 4. The Category of Abelian $p$-groups

The category of Abelian $p$-groups is Abelian with a generator and exact direct limits. If $U$ is a generator in this category, the trace ideal of $U$ in $E$ is 0 (Theorem 4.2) and $U$ is flat as a module over $E$ (Theorem 4.1). It is not known which generators are proper, but some information about them is provided in 4.4, 4.5 and 4.6. The reader is referred to [2] for the group theoretical facts used in the sequel. The category of Abelian $p$-groups will be denoted by $\mathcal{E}$.

**Theorem 4.1.** Let $G$ be an Abelian $p$-group with maximum divisible subgroup $D$. If $G/D$ is unbounded, then $G$ is flat as a module over its endomorphism ring.

**Proof.** It is easy to see that the category $\mathcal{E}$ of Abelian $p$-groups is Abelian with exact direct limits. Also, any $G \in \mathcal{E}$ with $G/D$ unbounded
is a generator. Let $E = \text{Hom}_G(G, G)$. It is easy to see that the adjoint of the functor $\text{Hom}_G(G, \cdot) : \mathcal{G} \to \mathcal{G}$ is just $(\otimes_G G) : \mathcal{G} \to \mathcal{G}$. By Theorem (5), this functor is exact. In other words, $G$ is flat as a module over $E$.

A classical result is that any Abelian $p$-group is determined by its endomorphism ring. The usual construction of a $p$-group $G$ from its endomorphism ring gives no hint that if $G/D$ is unbounded, then $G$ is $E$-flat. However 4.1 suggests that if $G/D$ is unbounded, then $G$ may be constructed from $E$ as a direct limit of projectives, showing at once that $G$ is determined by $E$ and is $E$-flat. This is indeed the case, and that construction has been carried out in another paper [6].

**Proposition 4.2.** Let $\mathcal{G}$ be the category of Abelian $p$-groups for some prime $p$. Let $U$ be a generator in $\mathcal{G}$. Let $U$ be the trace ideal of $U$ in $E = \text{Hom}_G(U, U)$, and let $I = \cap \{ I \mid I$ is a right ideal of $E$. $IU = U \}$, then $I = I' = 0$.

**Proof.** Let $x \in U$, $x \neq 0$, $o(x) = p^k$ and let $m$ be any positive integer. Since $U$ is a generator, $U$ has an unbounded basic subgroup, so $U$ has a cyclic summand $Z_y$ of order $\geq p^{m+n}$. If $U = Z_y \oplus H$, let $z : U \to U$ be the homomorphism defined by $z(y + h) = rx$ for all $x \in E$, $h \in H$. Write $r = p^m t$ with $(r, p) = 1$. If $x^e = 0$, then $k < n$ and $o(xz) = p^{m-k}$. Now $o(xz + h) \geq o(xz) = p^{m+n-k} \geq p^k \geq o(x)(r + h)$. In fact for $u \in U$ with $o(u) = 0$, $u = p^k t y + h$ and

\[ o(x) - o(xz) \geq p^{m+n-k} \geq p^{m-k} \geq p^{m} - 1. \]

Now let $I(m) = \{ u \in E \mid o(u) = o(xz) \geq p^{m} - 1 \}$ if $o(u) > 0$. Then $I(m)$ is a right ideal of $E$, for $e \in E$, $a \in I(m)$ and $a(xz) > 0$ then $o(u) = o(a(xz)) \geq o(a(u)) \geq o(xz) \geq p^{m} - 1$. From the first paragraph it is clear that $I(m)U = U$ for any $m > 0$. Thus $I \subseteq I' \subseteq I(m) = 0$.

**Corollary 4.3.** $\text{Hom}_E(U, E) = 0$.

The remainder of this section is an attempt to determine the proper generators in the category of Abelian $p$-groups. First it is shown that not every generator is proper. A $p$-group is torsion-complete if it is the torsion-subgroup of its closure in the $p$-adic topology.

**Proposition 4.4.** Let $G$ be an unbounded torsion-complete Abelian $p$-group. Then $G$ is a generator in the category of Abelian $p$-groups, but not proper.

**Proof.** The group $G$ is a generator since it is unbounded and reduced. Let $B$ be a basic subgroup of $G$. The pure exact sequence

\[ 0 \to B \to G \rightarrow G/B \rightarrow 0 \]
yields the exact sequence

\[ 0 \rightarrow \text{Hom}(G, B, G) \rightarrow \text{Hom}(G, G) \rightarrow \text{Hom}(B, G) \rightarrow \text{Pext}(G/B, G). \]

(Pext(G/B, G) is the group of pure extensions of G by G/B. See [5].) Now \( \text{Hom}(G/B, G) = 0 \) since G/B is divisible and G is reduced. Also \( \text{Pext}(G/B, G) = 0 \) since G is torsion-complete. Then \( \text{Hom}(G, G) \rightarrow \text{Hom}(B, G) \) is an isomorphism. But B + G since B is a direct sum of cyclic groups and G is not. Hence G is not proper.

**Proposition 4.5.** Every non-reduced generator in the category of Abelian p-groups is a proper generator.

**Proof.** A non-reduced p-group G has \( Z(p^\infty) \) as a summand, and \( Z(p^\infty) \) is a co-generator, whence G is proper.

**Theorem 4.6.** Let \( H \) be a generator in the category of Abelian p-groups. If \( \text{Pext}(G, G) = 0 \) and G is not torsion-complete, then G is proper.

**Proof.** By 4.4, G may be assumed to be reduced. Suppose \( H \subset G \), \( H + G \), and \( \text{Hom}(G, G) \xrightarrow{\Phi} \text{Hom}(H, G) \) is an isomorphism. Then \( G/H \) is divisible, since otherwise \( \text{Hom}(G/H, G) = 0 \) and \( \Phi \) would not be a monomorphism. Assume that \( H \) is not pure in G. Then no basic subgroup of \( H \) is pure in G, and so \( H \) has a cyclic summand, generated by \( \ell \), say, that is not a summand of G. If \( \ell \) is of order \( p^n \), then \( p^{n-1} \ell \) is of height at least \( n \) in G. That is, there is a \( g \in G \) such that \( p^ng = p^{n-1} \ell \). Project \( H \) onto the cyclic summand \( aH \) generated by \( \ell \). Suppose this projection can be extended to an endomorphism \( \alpha \) of \( G \). Then the order of \( \alpha(\ell) \) is \( p^{n+1} \) and so \( \alpha(\ell)/aH = 0 \). However, since \( G/H \) is divisible, \( \alpha(\ell)/aH \) is divisible. But \( aH \) is finite and \( \alpha(\ell) \) is reduced. This is impossible.

Thus \( H \) may be assumed to be pure in G, and the exact sequence \( 0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0 \) yields the exact sequence

\[ \text{Hom}(G, G) \xrightarrow{\alpha} \text{Hom}(H, G) \rightarrow \text{Pext}(G/H, G) \rightarrow \text{Pext}(G, G) = 0. \]

Since G is not torsion-complete, \( \text{Pext}(G/H, G) = 0 \), so \( \Phi \) is not an isomorphism. This concludes the proof.

One justification for Theorem 4.5 is

**Corollary 4.7.** Let \( G \) be a generator in the category of Abelian p-groups. If \( G \) is a direct sum of cyclic groups, then \( G \) is a proper generator.

**Proof.** \( G \) is not torsion-complete, being an unbounded direct sum of cyclic groups. Since \( \text{Pext}(G, G) = 0 \), 4.5 applies.

References


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