SPLITTING PROPERTIES OF HICK SUBGROUPS:

by

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In this paper we continue the investigation of high subgroups of Abelian groups. (See [3], [4] and [5].) All groups considered here will be Abelian. If \( G \) is a group, \( G' \) denotes the subgroup of elements of infinite height in \( G \left( G' = \bigcap_{n=1}^{\infty} (nG) \right) \), and \( G_1 \) denotes the torsion subgroup of \( G \).

A high subgroup of \( G \) is any subgroup of \( G \) maximal disjoint from \( G' \). A group \( G \)-split if \( G \) is a summand of \( G' \). In general, we adopt the notation used in [1].

One of the fundamental problems in Abelian group theory is to find decent necessary and sufficient conditions for a group to split. We investigate here the relation between the splitting of a group \( G \) and the splitting of high subgroups of \( G \).

Our main result states (Theorem 2) that a reduced group \( G \) splits if and only if \( G/G_1 \) is reduced and some high subgroup of \( G \) splits. In a sense, this reduces the splitting problem for arbitrary groups to groups with no elements of infinite height.

Lemma. — If \( H \) is a high subgroup of \( G \), then \( H/H_1 \) is a summand of \( G/H_1 \).

Proof. — By [3], \( H_1 \) is high in \( G \), and \( G/H_1 \) is divisible. Since a divisible subgroup is an absolute direct summand, \( G/H_1 = G/H_1 \oplus R/H_1 \), with \( R \subseteq R \). Let

\[
\frac{H}{H_1} = D/H_1 \oplus F/H_1,
\]

(*) This research was supported by N.S.F. grant G 17/976.
where $D/H_i$ is divisible and $F'/H_i$ is reduced. Now we may write

$$R/H_i = E/H_i \oplus D/H_i \oplus S/H_i,$$

where $E/H_i \oplus D/H_i$ is divisible, $S/H_i$ is reduced, and $F \leq S$. Now

$$D + F = H \leq D + S.$$

Assume $(d + x) \in G$, $d \in D_i$, $x \in S$. Then

$$(d + x) + H \leq (D + S)/H_i = (D/H_i) \oplus (S/H_i) = D/H_i,$$

since $S/H_i$ is a reduced subgroup of the torsion-free group $R/H_i$. Hence $(d + x) \in D \cap G \leq H \cap G = 0$. Since $H$ is high in $G$, we get $H = D + S$, and it follows that $S \cong F$. Therefore

$$H/H_i = D/H_i \oplus S/H_i,$$

and so

$$G/H_i = G_i/H_i \oplus E/H_i \oplus H/H_i.$$

**Theorem 1.** Let $H$ be a high subgroup of $G$, and suppose $H = H_i \oplus L$. Then $G = M \oplus L$, where $M/G_i$ is the divisible part of $G/G_i$.

**Proof.** From the lemma, we have that

$$G/H_i = G_i/H_i \oplus E/H_i \oplus H/H_i,$$

with $G_i/H_i \oplus E/H_i$ divisible. Let $M = G_i + E$. Then

$$G/H_i = M/H_i \oplus (H_i \oplus L)/H_i,$$

and hence $G = M \oplus L$ by [6], lemma 6. Now

$$G/G_i = M/G_i \oplus (L \oplus G_i)/G_i.$$

Since $M/H_i = G_i/H_i \oplus E/H_i$ is divisible, $M/G_i$ is divisible.

But $(L \oplus G_i)/G_i \cong L$ is reduced, so that $M/G_i$ is the divisible part of $G/G_i$.

It is interesting to note that $M$ is the only summand of $G$ complementary to $L$. In fact, if

$$G \cong N \oplus L = M \oplus L,$$

then $G_i \cong N$, and

$$G/G_i = N/G_i \oplus (L \oplus G_i)/G_i \cong M/G_i \oplus (L \oplus G_i)/G_i.$$

Since $M/G_i$ is the divisible part of $G/G_i$, $N/G_i$ must also be the divisible part of $G/G_i$, and hence $N = M$.

In trying to determine necessary and sufficient conditions for a group $G$ to split, one may as well assume $G$ is reduced. If $G$ is reduced, then a necessary condition for $H$ to split is that $G/H$ be reduced. Examples are
easy to find which show that this condition is not sufficient. However, from the previous theorem we obtain the following interesting necessary and sufficient condition that a reduced group split.

**Theorem 2.** — Let $G$ be reduced. Then $G$ splits if and only if $G/G_i$ is reduced and some high subgroup of $G$ splits.

**Proof.** — Suppose $G$ splits. Let $G = G_0 * L$, and let $H_i$ be a high subgroup of $G_i$. Now $L$ is torsion free reduced, so that $H_i * L$ has no elements of infinite height in $G$. It follows readily that $H_i * L$ is high in $G$, and hence that $G$ has high subgroup that splits. Since $G/G_i \cong L$, $G/G_i$ is reduced.

Suppose $G/G_i$ is reduced and that $G$ has a high subgroup $H$ that splits. Let $H = H_i * L$. From the previous theorem we have $G = G_i * L$, where $G_i$ is the divisible part of $G/G_i$. But $G/G_i$, is reduced, and hence $G \cong G_i * L$ and $G$ splits.

In theorems 2 the hypothesis that $G/G_i$ be reduced is required, and we now give an example to show this. Suppose $G$ is a reduced group with the properties:

1. $G/G_i \cong Q$, the group of rational numbers;
2. $G^i$ is torsion free and not zero.

Let $H$ be a high subgroup of $G$. Then since $H_i$ is high in $G_i$ and $(G_i)^{G_i} = e$, $H_i = G_i$. Thus, from the lemma, $H/G_i$ is a summand of $G/G_i$. This means $H/G_i = 0$ since $H \neq G$ and $G/G_i$ is indecomposable. Thus $H = G_i$ is the only high subgroup of $G$. $G$ does not split since $G$ is reduced and $G/G_i$ is divisible, but $G_i$, the only high subgroup of $G$, does split trivially.

To demonstrate the existence of such a group we employ homological methods and results. Let $B$ be an unbounded closed $p$-group. (See [1], p. 114.) Let $B$ be a basic subgroup of $B$. We may write $B = \bigoplus_{x \in A} (C_x/B)$, where $C_x/B \cong Z(p^x)$ for all $x$. Let $C = \bigoplus_{x \in A} (C_x/B)$ Then

$$B = \bigoplus_{x \in A} (C_x/B),$$

so that $C$ is pure in $B$ and $B/C \cong Z(p^\infty)$. Let $Q$ denote the group of rational numbers and $Z$ the group of integers. $G = \text{Ext}(Q/Z, C)$ is a reduced group such that $G \cong C$ and $G/G_i$ is divisible and not $0$. (See [2], p. 370-376.) Since $C$ is a $p$-group, $\text{Ext}(Z(q^\infty), C) = 0$ for all primes $q \neq p$. Thus, if $P = \text{set}$ of all primes,

$$G = \text{Ext}(Q/Z, C) = \text{Ext} \left( \bigoplus_{q \in P} Z(q^\infty), C \right) \cong \prod_{q \in P} \text{Ext}(Z(q^\infty), C) \cong \text{Ext}(Z(p^\infty), C).$$
The pure exact sequence \( 0 \to C \to \tilde{B} \to Z(p^\infty) \to 0 \) yields the exact sequence

\[
\text{Hom}(Z(p^\infty), \tilde{B}) \to \text{Hom}(Z(p^\infty), Z(p^\infty)) \\
\to \text{Ext}(Z(p^\infty), C) \to \text{Ext}(Z(p^\infty), \tilde{B}).
\]

But \( \text{Hom}(Z(p^\infty), \tilde{B}) = 0 \) since \( \tilde{B} \) is reduced and \( Z(p^\infty) \) is divisible, and \( \text{Ext}(Z(p^\infty), \tilde{B}) = 0 \) since \( \tilde{B} \) is a closed group and \( Z(p^\infty) \) is torsion. (See [1], p. 117.) Thus from the exactness of the above sequence,

\[
\text{Ext}(Z(p^\infty), C) \cong \text{Hom}(Z(p^\infty), Z(p^\infty)).
\]

This is the group of \( p \)-adic integers, (see [1], p. 311), so it is torsion free and not zero. Moreover \( \text{Ext}(Z(p^\infty), C) \cong C^* \). (See [1], p. 311.) Thus we have a group \( G \) such that \( G / G_i \) is divisible and not zero and \( G \) is torsion free and not zero.

Now let \( g \) be a non-zero element of infinite height in \( G \). Since \( G / G_i \) is torsion free divisible we may write

\[
G / G_i = A / G_i \oplus B / G_i,
\]

where \( g \in A \) and \( A / G_i \cong Q \). \( A \) is pure in \( G \) so \( A^* = A \cap G^* \neq 0 \). The group \( A \) has the desired properties.

It is not known whether or not high subgroups of torsion groups \( T \) are endomorphic images of \( T \). The group \( G : = \text{Ext}(C / Z, C) \) constructed above is an example of a mixed group such that none of its high subgroups are endomorphic images. This group is cotorsion, i.e., \( \text{Ext}(A, G) = 0 \) for all torsion free groups \( A \) and \( G \) is reduced. (See [2].) A homomorphic image of a cotorsion group is the direct sum of a cotorsion group and a divisible group. (See [7].) Thus if a high subgroup of \( G \) is an endomorphic image, it must be cotorsion. Any high subgroup \( H \) of \( G \) contains \( G_i \), and is pure so that \( G / H \) is torsion free divisible, and not zero. Let \( Q \) denote the group of rational numbers. Then the exact sequence

\[
0 \to H \to G \to G / H \to 0
\]

yields the exact sequence

\[
0 \to \text{Hom}(C / G, H) \to \text{Ext}(C / H, H) \to 0,
\]

\( G \) being cotorsion. But \( \text{Hom}(C / G, H) \neq 0 \), and hence \( H \) is not cotorsion. Thus \( H \) is not an endomorphic image of \( G \).

It would be interesting to know the class of groups whose high subgroups are endomorphic images. Torsion groups whose high subgroups have this property include those torsion groups whose high subgroups are direct sums of cyclic groups. (See [3].) If every high subgroup of a group \( G \) is a direct sum of cyclic groups, is every high subgroup of \( G \) an endomorphic image of \( G \)?
We proceed now to discuss high subgroups in reduced groups $G$ that split. First we give a characterization of the high subgroups of such groups.

**Theorem 3.** — Let $G := G_i \oplus S$, with $G$ reduced. Then there is a one-to-one correspondence between the set of all high subgroups of $G$ and the set $\bigcup_{\gamma \in \Gamma} \text{Hom}(S, G_i/K_{\gamma})$, where $\{ K_{\gamma} \}_{\gamma \in \Gamma}$ is the set of all high subgroups of $G_i$.

**Proof.** — Let $K_{\gamma}$ be high in $G_i$, and let $x \in \text{Hom}(S, G_i/K_{\gamma})$. Then $x$ induces an isomorphism $\delta$ from $S/Ker(x)$ onto $T/K_{\gamma}$, where $K_{\gamma} \leq T \leq G_i$. Let

$$K := \{ t + s \mid (s + \text{Ker}(x)) \delta = t + K_{\gamma} \}.$$ 

We show that $K$ is high in $G$. Since $G$ is reduced, $L$ is torsion free reduced, and hence $G^1 := (G_i)^1$. Since $\pi \cap G = K_{\gamma}$, we have that $K \cap G^1 = 0$. Suppose $g \in K$. Then $g = g_i - s, g_i \in G_i, s \in S$, and $g_i \in K_i$. There exists $t \in T$ such that $(t + s) \in K$. Hence $(g_i + s) = (g_i + t) \in K_i$.

Since $K_{\gamma}$ is high in $G_i$,

$$a \in \langle g_i - t, K_{\gamma} \rangle \cap G^1 \leq \langle g, K \rangle \cap G^1,$$

whence $K$ is high in $G$. Clearly distinct $x$'s in $\bigcup_{\gamma \in \Gamma} \text{Hom}(S, G_i/K_{\gamma})$ give rise to distinct high subgroups.

Now let $K$ be high in $G := G_i \oplus S$. Then $K_i$ is high in $G_i$. (See [3].) Suppose $s \in S, t \in K$. Then

$$a \in \langle ns + k, K_i \rangle \leq (G_i)^1$$

for some integer $n$ and $k \in K_i$. Since $g_i \in G_i$, $g_i = ng_i, g_i \in G_i$, and from the parity of $K$ (see [3]), we have $k = nk_i$ with $k_i \in K_i$. Thus

$$s + k_i - g_i = g_i \in G_i.$$ 

Hence the group of $S$ components of the elements of $K$ is $S$. Let $T$ be the group of $G_i$ components of the elements of $K$. Then $K \leq T \leq G_i$, $K$ is a subdirect sum of $T$ and $S$, and

$$S/(K \cap S) \cong T/(K \cap T) \cong T/K_i.$$ 

The theorem follows.

A natural question to ask is the following. If one high subgroup of a group $G$ splits, do all high subgroups of $G$ split? The answer is negative, even if $G$ itself splits, as the following example shows.
EXAMPLE. — Let $G$ be any reduced $p$-group with non-zero elements of infinite height. (e. g., let $G$ be the Prüfer group. See [1], p. 105.) Let $S$ be the group of rational numbers with denominators powers of $p$, and let $G = G \oplus S$. Let $H$ be high in $G$. Since $G/H$ is divisible there exists a subgroup $T$ of $G$ such that $T/H \cong \mathbb{Z}/(p^n)$. Let $R$ be any subgroup of $S$ such that $S/R \cong \mathbb{Z}/(p^n)$. (Let $R$ be the integers for example.) Let $\delta$ be an isomorphism from $T/H$ onto $S/R$. From the proof of theorem 3, we have that $H = \{ t + s \mid (t + H) \delta = s + R \}$ is a high subgroup of $G$, and $H_\delta$ is the torsion subgroup of $H$. Suppose $H$ splits and $H = H_\delta \oplus V$. Now $V$ is a subdirect sum of $T_\delta$ and $S$ (see proof of theorem 3), where $H_\delta \leq T_\delta \leq T$. Furthermore $T_\delta/(T_\delta \cap V) \cong S/(S \cap V)$. But $V$ is torsion free so that $T_\delta \cap V = 0$. Hence $T_\delta$ is a homomorphic image of $S$. But $T_\delta$ is reduced and the only reduced $p$-group that is a homomorphic image of $S$ is $0$. Therefore $T_\delta = 0$, and so $0 = H_\delta \leq T_\delta$. But no high subgroup of $T$ is $0$. This contradiction establishes that $H$ does not split.

However, $G$ does have a high subgroup that splits, namely $H \oplus S$. (See theorem 3.) In particular, we have that two high subgroups of a group are not necessarily isomorphic. It is still not known whether or not two high subgroups of a torsion group are isomorphic.

Although two high subgroups $H$ and $K$ of a group $G$ are not necessarily isomorphic, and in fact $H$ may split and $K$ not split, it is true that high subgroups do provide several invariants, in the following sense.

**Theorem 3.** — Let $H$ and $K$ be high subgroups of $G$. Then

(a) \[ G/H \cong G/K; \]

(b) \[ H/K \cong K/K_\delta; \]

(c) \[ G/H \cong G/K. \]

**Proof.** — The proof of (a) may be found in [3].

If $A$ is a subgroup of $G$, let $\bar{A} = (A + G)/G$. Then $\bar{G}$ is maximal disjoint from $\bar{G}$ in $\bar{G}$. To see that $\bar{H} \cup \bar{G} = 0$, suppose that $h + G = g + G$, with $h \in H$, $g \in G$. Then from $h - g \in G$, it follows that for some integer $m$, $mh = mg = 0$, whence $h + G = 0$. Next suppose there exists $g + G \in \bar{H}$, and that $g + G \in \bar{G} = 0$. Since $H$ is high in $G$, there exists $h \in H$, $g \in G$ such that $mh + mg = mg$. Now $g \in G$, and hence $g \in G$. Thus $h + g - g = h \in G$.

and so

$g + G = h_1 + (g_1 + g_1) + G = - h + G \in \bar{H}$.  

But $g + G \in \bar{H}$. We conclude that $\bar{H}$ is maximal disjoint from $\bar{G}$. Since $\bar{G}$ contains a (unique) minimal divisible
subgroup \( \bar{B} \) which contains \( \bar{G} \) and \( \bar{B} \cap \bar{D} = \varnothing \). (See [3].) But \( \bar{B} \) is an absolute summand so that \( \bar{B} \cong \bar{H} \oplus \bar{D} \), for any high subgroup \( H \) of \( G \).

Thus \( \bar{B} \cong \bar{K} \). Finally,

\[
H/H_i := H/(H \cap G_i) \cong (H \cap G_i)/G_i \cong \bar{H} \cong \bar{K} \cong K/K_i,
\]

and (6) is proved.

To prove (c), first notice that

\[
G/H_i \cong G_i/H_i \oplus R/H_i, \quad \text{and} \quad G/K_i \cong G \oplus S/H_i.
\]

Since \( H_i \) and \( K_i \) are high in \( G_i \) (see [5]), by (a), \( G_i/H_i \cong G_i/K_i \). Also \( R/H_i \cong G_i/G_i \cong S/K_i \). Hence \( G/H_i \cong G/K_i \), as stated.

We remark that in the case where \( H/H_i \) is reduced for some high subgroup \( H \) of \( G \), and in particular when some high subgroup of \( G \) splits, that all high subgroups of \( G \) may be obtained as follows. Let \( K_i \) be high in \( G_i \) and \( K/K_i \) high in \( G/K_i \). Then \( K \) is high in \( G \) and every high subgroup of \( G \) with torsion subgroup \( K \), is such a \( K \).

In [5] the notion of \( \Sigma \)-group was introduced. A group \( G \) is a \( \Sigma \)-group if and only if each high subgroup of \( G \) is a direct sum of cyclic groups. As a corollary to the preceding theorem we obtain the following result, proved in the torsion case in [5] and in the general case by Paul Hill (Abstract 582-57, Notices of the American Mathematical Society, August 1961).

**Corollary.** — If one high subgroup of a group \( G \) is a direct sum of cyclic groups, then \( G \) is a \( \Sigma \)-group, and any two high subgroups of \( G \) are isomorphic.

**Proof.** — Let \( H \) and \( K \) be high subgroups of \( G \) with \( H \) a direct sum of cyclic groups. Since \( H_i \) and \( K_i \) are high in \( G_i \), we have by theorem 7 in [5], that \( K_i \) is a direct sum of cyclic groups and \( H_i \cong K_i \). But \( H/H_i \cong K/K_i \), is free so that \( K \) is a direct sum of cyclic groups and \( H \cong K \).

Clearly \( \Sigma \)-groups form a class of groups in which all high subgroups split and are isomorphic. However the class of such groups properly contains the class of all \( \Sigma \)-groups. In fact, let \( P \) be the Prüfer group for the prime \( p \). (See [1], p. 105.) Let \( S \) be the group of rational numbers with denominators a power of the prime \( p \). Then every high subgroup of \( G = P \oplus S \) splits and any two high subgroups of \( G \) are isomorphic. We merely outline the proof of this fact. Let \( H \) be high in \( G \), and let \( P_H \) be the group of \( P \) components of the elements of \( H \). Then \( H \) is a subdirect sum of \( P_H \) and \( S \), and

\[
P_H/(H \cap P) \cong P_H/H \cong S/H \cap S.
\]
Thus \( S/(H \cap S) \) is a \( p \)-group, and if \( n \) is the smallest positive integer such that \( s^{p^n} \notin H \cap S \), then

\[
S/(H \cap S) = \langle s^{p^n} + H \cap S \rangle \cong \mathbb{Z}/p^k \mathbb{Z}
\]

is finite cyclic. Since \( H_c \) is pure in \( P_n \), we have that

\[
P_n = H_c \oplus H_c.
\]

Thus

\[
H \leq H_c \oplus R \oplus S, \quad H = H_c \oplus (H_c \cap (R \oplus S)),
\]

and so \( H \) splits. Since \( H/H_c \cong S \), we have \( H \cong H_c \oplus S \). Any subgroup high in \( P \) is a basic subgroup of \( P \) (see [1], p. 98), whence any two high subgroups of \( P \) are isomorphic. It follows that every high subgroup of \( G \) is isomorphic to \( H_c \oplus S \).

It would be interesting to know the class of groups in which all high subgroups split, the class of groups in which all high subgroups are isomorphic, and the intersection of these two classes.

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(Manuscript received le 20 août 1961.)

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