TORSION ENDOMORPHIC IMAGES OF MIXED ABELIAN GROUPS

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In this paper we will answer Fuchs' PROBLEM 32 (a), and the corresponding part of his PROBLEM 33. (See [1], pg. 290.) The statement of these PROBLEMS are the following.

I. "Which are the torsion groups $T$ that are endomorphically images of all groups containing them as maximal torsion subgroups?"

II. "Which are the torsion groups $T$ such that a basic subgroup of $T$ is an endomorphically image of any group $G$ containing $T$ as its maximal torsion subgroup?"

Actually, we will answer question II and the following question which is more general than I.

III. What groups $Y$ are endomorphically images of all groups $G$ containing $H$ such that $G/H$ is torsion free?

The solution will be effected by using some homological results of Harrison [2]. All groups considered here will be Abelian. The definitions and results stated in the remainder of this paragraph are due to Harrison, and may be found in [2]. A reduced group $G$ is cotorsion if $\text{Ext}(A, G) = 0$ for all torsion free groups $A$. If $H$ is a reduced group, then $\text{Ext}(Q/Z, H) = H'$ is cotorsion, where $Q$ and $Z$ denote the additive group of rationals and integers, respectively. Furthermore, $H$ is a subgroup of $H'$ (that is, there is a natural isomorphism of $H$ into $H'$) and $H'/H$ is divisible torsion free. This implies, of course, that if $T$ is a torsion reduced group, then $T$ is the torsion subgroup of $T' = \text{Ext}(Q/Z, T)$.

Now it is easy to see that if $G$ is a group such that $\text{Ext}(A, G) = 0$ for all torsion free groups $A$, then any homomorphic image of $G$ is the direct sum of a cotorsion group and a divisible group. In fact, let $H$ be a homomorphic image of $G$. This gives us an exact sequence

$$0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$$

which yields the exact sequence

$$0 \rightarrow \text{Hom}(A, K) \rightarrow \text{Hom}(A, G) \rightarrow \text{Hom}(A, H) \rightarrow \text{Ext}(A, K) \rightarrow \text{Ext}(A, G) \rightarrow \text{Ext}(A, H) \rightarrow 0.$$ 

If $A$ is any torsion free group, then $\text{Ext}(A, G) = 0$, and so $\text{Ext}(A, H) = 0$. Write $H = D \oplus L$, where $D$ is the divisible part of $H$. Then $L$ is reduced, and $0 = \text{Ext}(A, D) \oplus \text{Ext}(A, L) = \text{Ext}(A, D) \oplus \text{Ext}(A, L)$, so that $L$ is cotorsion. Our assertion is proved.

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Now we are ready to give the solutions promised earlier. The following theorem settles III.

**Theorem.** The group $H$ is an endomorphic image of every group $G$ containing it such that $G/H$ is torsion free if and only if $H = D \oplus C$, where $D$ is divisible and $C$ is cotorsion. This is equivalent to the assertion that $H$ is a direct summand of every such $G$.

**Proof.** Suppose $H$ is an endomorphic image of every group $G$ containing it such that $G/H$ is torsion free. Let $H = D \oplus C$, where $D$ is divisible and $C$ is reduced. Then $C$ is a subgroup of the cotorsion group $\text{Ext}(Q/Z, C) \cong C'$ such that $C/C'$ is torsion free, so that $H$ is a subgroup of $D \oplus C' = H'$ such that $H'/H$ is torsion free. Therefore $H$ is an endomorphic image of $H'$. Let $\text{Ext}(A, D \oplus C') = 0$ for all torsion free groups $A$, and as we have just proved, any homomorphic image of $D \oplus C'$ is the direct sum of a cotorsion and a divisible group. It follows that $C'$ must be cotorsion.

If $H = D \oplus C$, with $D$ divisible and $C$ cotorsion, then $\text{Ext}(A, H) = 0$ for all torsion free groups $A$, and hence $H$ is a direct summand of any group $G$ containing it such that $G/H$ is torsion free. If $H$ is a direct summand of any such $G$, then clearly $H$ is an endomorphic image of any such $G$. Thus our theorem is proved.

The torsion group $T$ is a direct summand of every group containing it as its maximal torsion subgroup if and only if $T = D \oplus B$, with $D$ divisible and $B$ of bounded order. (See [1], pg. 187.) Thus, by our theorem, we see that the torsion group $T$ is an endomorphic image of every group containing it as its maximal torsion subgroup if and only if $T = D \oplus B$, with $D$ divisible and $B$ of bounded order.

The solution of II goes as follows. Suppose a basic subgroup of $T$ is an endomorphic image of every group $G$ in which $T$ is the maximal torsion subgroup. Let $T = D \oplus B$, with $D$ divisible and $B$ reduced. Then a basic subgroup of $T$ must be an endomorphic image of $B \oplus B' = D \oplus \text{Ext}(Q/Z, B)$. Therefore a basic subgroup of $T$ must be cotorsion, since it is reduced, and since it is torsion, it is of bounded order. (See [1], pg. 187. The remark by Harrison in [2], pg. 371 is incorrectly worded.) Writing $T$ as $D \oplus B$, we see that a basic subgroup of $B$ is a basic subgroup of $T$. But any two basic subgroups of $T$ are isomorphic, and if $B$ has a basic subgroup of bounded order, then $B$ must be of bounded order. In fact, the only basic subgroup of $B$ is $B$ itself. Thus $T = D \oplus B$, with $D$ divisible and $B$ of bounded order. If $T = D \oplus B$, with $D$ divisible and $B$ of bounded order, then $T$ is a basic subgroup of $T$. Now $D \oplus B$, and hence $B$, is a direct summand of any $G$ in which $T$ is the maximal torsion subgroup. Therefore $B$ is an endomorphic image of any such $G$, and hence any basic subgroup of $T$. 
is such an endomorphic image. Thus we see that the answers to questions I and II are the same.

REFERENCES


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