Abelian Group Theory
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1. Introduction. Fix a prime \( p \) and denote the ring of integers localized at \( p \) by \( \mathbb{Z}_p \). Throughout, all modules will be \( \mathbb{Z}_p \)-modules. Of course, our use of \( \mathbb{Z}_p \)-modules is just a convenient device for dealing with \( p \)-local abelian groups, that is, abelian groups for which multiplication by each prime \( q \neq p \) is an automorphism.

In [13], Warfield studied one class of modules which arises as summons of simply presented modules. This paper is a survey of the existing theory of such modules (which we call Warfield modules) together with a number of new results. The classification of these modules in terms of numerical invariants represents the most recently completed stage in a natural progression which began with the work of Ulm and Ulm on countable \( p \)-groups and went through successive generalizations to direct sums of countable \( p \)-groups and simply presented (also called totally projective) \( p \)-groups.

The central theme throughout is the notion of height - both the Ulm and Warfield invariants are defined in terms of height, and height concepts are at the heart of nearly all the proofs. For this reason, we have taken the valued viewpoint, creating height as an entity apart from a module, thereby separating and emphasizing its role. We have also taken a different approach from that of Warfield in a number of respects and because of this, provide new proofs of many known theorems. Perhaps the most essential difference is our starting point - Warfield modules are defined as extensions in the category of valued modules, of direct sums of cyclic by simply presented torsion modules. This definition suggests that the theory of such modules will include both the theory of completely decomposable torsion free modules (those correspond to special direct sums of cyclic valued modules) and the rather extensive theory of simply presented torsion modules. This is indeed the case. There is an isomorphism theorem which tells us, in terms of numerical invariants, when two Warfield modules are...
lemmata, and an existence theorem giving necessary and sufficient conditions for the existence of a Warfield module with prescribed numerical invariants. These two results are the main feature, and in combination enable us to prove many decomposability properties.

In many other respects, the theory of Warfield modules parallels the theory of singly presented torsion modules. Aside from the reasons already given, this is to be expected in view of the fact that Warfield modules arise by examining the various characterizations of singly presented torsion modules and trying to generalize them in some way. The most obvious approach, and indeed, the one used by Warfield, is to drop the torsion requirement from the definition of a singly presented torsion module. The problem is that, in general, summands of singly presented modules with elements of infinite order are not singly presented, and closure under taking summands is responsible for many of the nice properties of the class of singly presented torsion modules. Since Warfield includes summands of simply presented modules in his study, this closure was assured and the resulting class of modules turned out to be the correct generalization.

The dominant 'torsion free' feature of Warfield modules is their possession of a nice decomposition basis, and a result of this is that the pathology of torsion free groups is avoided. The need for a nice decomposition basis rather than just a decomposition basis allows the extension of maps from the basis to the containing module (of course, other conditions must also be satisfied).

With this in mind, we have included a fairly close analysis of the conditions which form a decomposition basis to be nice, or have a nice sub-basis. An important technique for dealing with modules having a decomposition basis is the construction of categories whose objects are modules and whose morphisms, in effect, ignore torsion. A Krull-Schmidt theorem is then proved in such a category and translated into a theorem about modules. Two categories of this type are the category $C$, discussed in detail in this paper, and the category $M$ which appears in [13]. The reason for introducing the category $M$ in [13] is to prove that a summand of a simply presented module has a nice decomposition basis. Until this is established, very little progress can be made, and, in particular, the
isomorphism theorem cannot be proved. On the other hand, our definition of a Warfield module gives us the nice decomposition basis for free and the isomorphism theorem follows directly. Of course, the difficulty is then to show closure under sums, and in particular that summands of simply presented modules are exactly our valued extensions. The category \( \Delta \) is introduced for just that purpose.

A brief outline of the paper follows. The word 'module' will always mean a valued module with the height valuation (see Section 2). This convention will be strictly observed and should be kept in mind at all times.

Section 1 outlines the basic concepts of valued tree and modules, all of which have been discussed previously in [4], [5] and [7]. In Section 3, \( \Delta \)- and derived \( \Delta \)-invariants are defined and the relationship between them explored in detail. As the name suggests, the derived \( \Delta \)-invariants arise from the right derived functor associated with the \( \Delta \)-invariants. Although derived \( \Delta \)-invariants vanish for modules, they are required in the proofs of the existence theorem [6] and the various decomposition theorems which follow from it.

Section 4 records some results concerning valued modules with composition series, these are just the torsion valued modules with homological dimension one in the category of valued modules. The modules with composition series are the simply presented torsion modules and we state the well-known isomorphism and existence theorems, due collectively to Hill [5], and crucially to Halin [4], which will be extended to Warfield modules in Sections 7 and 10 respectively.

In Section 5 we extend to valued modules an invariant defined by Warfield [16] for modules. The treatment here is a generalisation of that given by Section 14. This invariant (which we call the Warfield invariant) gives the essential information we require about the torsion-free structure of Warfield modules and it is shown that if a valued module has a decomposition basis, then that basis determines this invariant.

Section 6 consists the proof of a set theoretical lemma which is used to 'juggle' relative \( \Delta \)-invariants in the proof of the isomorphism theorem. A
special case of this lemma has appeared in the literature in various forms and references are given.

In Section 1, Warfield modules are defined as valued extensions and the isomorphism theorem is proved. Also included is a generalization of this theorem due to Stanton [12]. Sections 8 and 9 are in the main directed toward showing that our Warfield modules are indeed the summands of simply presented modules.

Section 8 introduces the category C which was discovered by Elbert Walker. A Krull-Schmidt theorem in C is shown to follow from the very general results of C. Walker and Warfield [15]. Some additional properties of this category are also given. In particular, it is shown that two valued modules A and B are C-isomorphic if and only if there are torsion valued modules S and T so that A ÷ S and B ÷ T are isomorphic as valued modules. Thus C provides the correct categorical setting for the notion of almost isomorphism introduced by Katsman and Yen [6]. For further discussion and applications of C to the theory of mixed groups, see the papers of Warfield and C. Yen in these proceedings.

In Section 5 it is shown that a summand of a Warfield module is Warfield and that, given a Warfield module A, there is a simply presented torsion module B so that A ÷ B is simply presented. It follows from these two results that the Warfield modules as defined in Section 7 are indeed the summands of direct sums of simply presented modules studied by Warfield in [12].

Section 10 deals with the existence and decomposition theorems for Warfield modules. Necessary and sufficient conditions for the existence of a Warfield module with prescribed Ulm and Warfield invariants were given in [4], and other results and examples from that paper are stated. The question of when a Warfield module is a direct sum of rank one modules is examined and a fairly satisfactory answer in terms of invariants is found. This is used to generalize a result of Katsman [9].

Section 11 is a study of decomposition bases and in particular, answers the question of whether a given decomposition basis necessarily has a nice subordinate. For example, it is shown that every countable decomposition basis has a nice subordinate, and this allows us to obtain the results of Warfield [16] concerning
countable modules with decomposition bases is in a more direct fashion. An example of a module with a decomposition basis which has no nice substructure is provided, and a result of Warfield [15, Lemma 4] is applied to show that such a module cannot have a nice decomposition basis.

Projective characterisations are discussed in Section 2, and the class of Warfield modules known as balanced projectives or X-modules, is scrutinised. An important result here is that any decomposition basis X of a balanced projective with the property that every element x of X has no gaps in its value sequence (such a basis is called an X-basis in [10]) generates a submodule <X> of A such that A/<X> is simply presented. This is proved by first showing that a balanced projective has homological dimension at most one in the category of valued modules.

In Section 3, we examine the torsion submodule of a Warfield module. It is shown that a Warfield module A can be decomposed A = B * C so that B is balanced projective and the torsion submodule of C is simply presented. In particular, the torsion submodule of A is an I-module in the sense of [10], and this part of the torsion submodule of A which is not simply presented lies in a summand (B above) which is simply presented and has a particularly simple structure. Thus there is a trade-off between the complexity of the torsion part of a Warfield module and the complexity of its torsion free structure.

We remark that [20] is a revised version of [15], and also contains a discussion of extensions of the theory of Warfield modules to a global setting. Another more recent discussion is also given in the paper of Warfield which appears in these proceedings.

### 2. Valued trees and modules

A tree is a set X, with a distinguished element 0, that admits a multiplication by p satisfying:

1) \( p0 = 0 \);

2) \( p^n x = x \) only if \( n = 0 \) or \( x = 0 \).

For a tree X and ordinal \( \alpha \), the subset \( p^\alpha X \) is defined inductively by setting \( p^0 X = X, p\alpha x = \{ px : x \in X \} \) and
\[ p^n x = \bigcup \{ p^m x : m \geq n \} \]

when \( n > 0 \). The height \( \kappa \) of an element \( x \) in \( X \) is \( \kappa \) if \( x \in p^n x \) for all ordinals \( n \), then we set \( \kappa = \infty \). The symbol \( \kappa \) satisfies \( \kappa > n \) and \( \kappa > n \) for all ordinals \( n \).

By a valued tree we mean a tree \( T \) together with a function \( v \) on \( T \) (called a valuation) such that:

1) \( v(x) \) is an ordinal or \( \infty \),
2) \( v(p^n x) > v(x) \).

A valued tree \( X \) is reduced if \( v(x) = \infty \) implies \( x = 0 \). The height function \( v \) is clearly a valuation satisfying \( v(x) > v(x) \) for all \( x \) in \( X \). Any tree is naturally valued by setting \( v(x) = v(x) \) for all \( x \). Conversely, if \( v(x) = v(x) \) for all \( x \), we say that the valued tree \( X \) is a tree. A map \( EF : X \rightarrow Y \) of valued trees is a function such that:

1) \( E(f(p)) = f(p') \),
2) \( v(f(x)) > v(x) \).

The resulting category is called the category of valued trees. An embedding of valued trees is a one-to-one map \( f \) such that \( v(f(x)) = v(x) \) for all \( x \). If the inclusion \( X \rightarrow Y \) is an embedding, we say that \( X \) is a valued subtree of \( Y \). If \( X \) is a valued tree, and \( \alpha \) is an ordinal, we set \( X(\alpha) = \{ x : v(x) = \alpha \} \).

A valued module is a module that is a valued cone and satisfies

\[ v(x + y) = \min(v(x), v(y)) \]

It follows that \( v(x) = v(x) \) if \( p \) does not divide \( n \). A map of valued modules is a module homomorphism that is a map of valued trees. A valued module which is a tree is called a valued module. The valued modules form a category \( V_p \). An embedding \( f : A \rightarrow B \) of valued modules is a map that is an embedding of valued trees. If the inclusion map \( A \rightarrow B \) is an embedding, we say that \( A \) is a valued submodule of \( B \).

The category \( V_p \) is pre-abelian (abelian with kernels and cokernels), and the theory of \([2]\) provides a natural definition of Ext. In \([7]\) it is shown that a sequence
of valued modules is in \( \text{Ext}(G, A) \) if and only if
\[ 0 = A(B) = B(A) = C(a) = 0 \]
is an exact sequence of modules for each \( a \). If \( A \subset B \) is an inclusion of valued modules, then an element \( b \) of \( B \) is \( A \)-\textit{regular} if \( B \) has maximal value among the elements in the coset \( b + A \). In case each coset of \( A \) contains an element of maximal value, \( A \) is said to be \textit{nice}. It is not difficult to see that \( G = A \subset B \subset C = 0 \) is in \( \text{Ext}(G, A) \) if and only if the inclusion \( A \subset B \) is a nice embedding.

We associate with each valued tree \( X \) a valued module \( S(X) \) in the following way. Let \( F_p = \mathbb{Q}(x)^p, (a) \) be the free \( \mathbb{Q} \)-module on the nonzero elements of \( x \). Let \( F_p \) be the submodule of \( F_p \) generated by the elements of the form \( p(x) \) where \( px = 0 \), and \( p(x) = (px) \) where \( px \neq 0 \),

and set \( S(x) = F_p / R_y \). Each element \( y \) of \( S(x) \) can be written in the form \( y = \sum u_i \cdot x \), where the \( u_i \)'s are units in \( F_p \). Setting \( v = \sum (v(x)) \) makes \( S(x) \) into a valued module. This valued module is called the \textit{simply presented valued module on} \( X \). If a valued module \( A \) is isomorphic to \( S(x) \), then the image of \( X \) in \( x \) is said to be a \textit{spanning tree} for \( A \). The notion of a spanning tree coincides with that of a T-\textit{basis} or a \textit{standard presentation} (see [1],[2],[14]). The usual definition of a simply presented module is a module that can be defined in terms of generators and relations so that the only relations are of the form \( px = 0 \) or \( px = y \). However, it is straightforward to prove (see, for example [7], Lemma 3.1) that such a module has a spanning tree, so there is no conflict. The \textit{rank} of \( A \subset F_p \) is the dimension of the vector space \( Q \) over the field of rational numbers \( Q \).

**Lemma 1.** A simply presented module is a direct sum of modules of rank at most one.

**Proof.** Let \( x \) be a spanning tree of \( A \). Define \( x, y \times x \) to be \textit{equivalent} if
there are positive integers $a$ and $b$ such that $p^a = q^b$, and set $(x_i)$ be the set of equivalence classes so obtained. Then it is easy to see that $\lambda = \sigma(x_i)$.

Observe that the functor $S$ is the adjoint of the forgetful functor from $V_\mathbf{p}$ to the category of valued trees. If $x \in Y$ is an ordering of valued trees, and $S(x)$ is the valued tree gotten from $Y$ by identifying $x$ with zero, then $x = S(x) = S(1) = S(Y) = 0$ is exact in $V_\mathbf{p}$.

Let $(A_i, \tau_i)$ be a family of valued modules such that $A_1$ has valuation $\tau_1$. Then the direct sum (coproduct) of the $A_i/x_i$ in $V_\mathbf{p}$ is readily seen to be the module direct sum with valuation $\tau = \min_i(\tau_i)$. If $x$ is a subset of a valued module $A$, then $\langle x \rangle$ denotes the valued submodule of $A$ generated by $x$. A valued module is cyclic if it is cyclic as a module. A basis of a direct sum of valued cyclics is $\mathbf{A}$ if a $\mathbf{c}$-independent subset $X$ of $\mathbf{A}$ such that $\alpha$ is the direct sum of $\langle \langle x_i \rangle \mid x_i \in X \rangle$. If $X$ and $Y$ are bases for $\mathbf{A}$, we say that $\mathbf{A}$ is subordinate to $\mathbf{Y}$ if every element of $X$ is a multiple of an element of $Y$.

A valued module $A$ is free if $\lambda$ is a direct sum of cyclics of infinite order with basis $X$ such that $\tau(x) = n = n + 1, 2, \ldots$

The value sequence of an element $a$ of a valued module $M$ is defined by $v(a) = \tau(a, \nu_1, \nu_2, \ldots)$. A value sequence is a sequence $\nu_1, \nu_2, \ldots$ of indices such that $\tau(x) = \nu_1$ for $x \in X, 1, \ldots$. Clearly, the value sequence of an element $a$ is a value sequence. If $s = a_1, a_2, \ldots$ is a value sequence, we write $v(s) = a_1, a_2, \ldots$ and if $s = \nu_1, \nu_2, \ldots$ we write $s = \nu_1$ to mean $a_1, a_2, \ldots$ for $n = 0, 1, \ldots$. Value sequences $\alpha$ and $\beta$ are said to be equivalent if there are positive integers $m$ and $n$ such that $a_n = n + 1$.

Observe that a cyclic valued module $A$ is determined up to isomorphism by the value sequence of any generator, and the value sequences of the elements of $A$ all lie in the same equivalence class.
If \( A \) has rank one then the value sequences of the elements of infinite order in \( A \) all lie in the same equivalence class.

Each submodule \( A \) of a module \( B \) is a valuation module with valuation given by restricting the height function on \( B \). This makes \( A \) a valuation submodule of \( B \). Conversely, each \( A \in \mathcal{V}_\mathcal{P} \) can be obtained this way:

**Theorem 2.** There is a functor \( T : \mathcal{V}_\mathcal{P} \to \mathcal{V}_\mathcal{P} \) such that, for each \( \mathcal{A} \),

1. \( TA \) is a module containing \( A \),
2. \( A \subseteq TA \) is a nice embedding,
3. \( TA/A \) is a simply presented torsion module.

The proof appears in [7].

3. **The invariant.** Let \( A \) be a valuation module and \( \alpha \) an ordinal.

Multiplication by \( p \) induces a natural map

\[
A(\alpha)/A(\alpha + 1) \to A(\alpha + 1)/A(\alpha + 2).
\]

We denote the kernel and cokernel of this map by \( F_A(\alpha) \) and \( G_A(\alpha + 1) \) respectively. Both \( F_A(\alpha) \) and \( G_A(\alpha + 1) \) are vector spaces over the \( p \)-element field. Clearly

\[
F_A(\alpha) = \{ z \in A(\alpha + 1) : p z \in A(\alpha + 2) \} / A(\alpha + 1)
\]

and

\[
G_A(\alpha + 1) = \{ A(\alpha + 1) \cap A(\alpha + 1) : p k(\alpha) \to \}
\]

The appropriate definition of \( G_A(\alpha) \) in general seems to be

\[
G_A(\alpha) = \frac{A(\alpha) - \{ A(\alpha + 1) : p k(\alpha) \}}{A(\alpha + 1) \cap A(\alpha + 1)}
\]

This definition agrees with the other for non-limit ordinals. The dimensions of \( F_A(\alpha) \) and \( G_A(\alpha) \) will be denoted \( \delta(\alpha) \) and \( \gamma(\alpha) \) respectively. We call

\[\delta(\alpha)\] the \( \alpha \)-th **invariant** of \( A \) and \( \gamma(\alpha) \) the \( \alpha \)-th **derived **
important of $A$. If $A$ is a module, $f_B(a)$ agrees with the usual definition of $U$-invariant while $f_B(a)$ vanishes for all $a$. On the other hand, there do exist valued modules $A$ which do not have the height valuation, and for which $f_B(a) = 0$ for all ordinals $a$ (for example, the valued $U$-submodules of the $p$-adic integers generated by $1$ and an irrational).

We now examine the connection between these two invariants. Let $0 = A + B + C + 0$ be exact in $V_p$. Then for each $a$,

$$0 = \frac{A(a)}{A(a + 1)} + \frac{B(a)}{B(a + 1)} + \frac{C(a)}{C(a + 1)} = 0$$

is an exact sequence of vector spaces. We arrive at the following commutative diagram:

$$0 \rightarrow F_A(a) \rightarrow F_B(a) \rightarrow F_C(a) \rightarrow 0$$

$$0 \rightarrow \frac{A(a)}{A(a + 1)} \rightarrow \frac{B(a)}{B(a + 1)} \rightarrow \frac{C(a)}{C(a + 1)} = 0$$

$$0 \rightarrow \frac{A(a + 1)}{A(a + 2)} \rightarrow \frac{B(a + 1)}{B(a + 2)} \rightarrow \frac{C(a + 1)}{C(a + 2)} = 0$$

$$0 \rightarrow f_A(a + 1) = f_B(a + 1) = f_C(a + 1) = 0.$$

The snake lemma applied to this diagram gives the exact sequence

$$0 \rightarrow f_A(a) \rightarrow f_B(a) \rightarrow f_C(a + 1) \rightarrow f_B(a + 1) = f_C(a + 1) = 0$$

This is the fundamental sequence relating the $F$'s and the $C$'s. Now $F_A$ is zero when $A$ is projective (free) in $V_p$, and $C$ is zero on modulus. Furthermore, there are enough projectives in $V_p$ (see [13]), and by Theorem 2 each $A \in V_p$ has a nice embedding in a module. Using this it is easy to show that $f_B(a)$ is the $n$th derived functor of $f_C(a + 1)$, and that $f_B(a + 1)$ is the right derived functor of $f_C(a)$.

For an arbitrary embedding $A \cong B$ (that is, we do not insist that $A$ be nice in $B$) the embedding $\delta : F_A(a) \rightarrow F_B(a)$ is still defined. The cokernel of $\delta$ is denoted $f_{BA}(a)$ and its dimension $f_B(a)$ is called the $a$-th $U$-invariant.
of \( n \) relative to 1, it is straightforward to check that if \( B \) is a module, then this definition agrees with the usual one. Notice that

\[ F_B(n) = F_B(n) \circ F_A(n). \]

There are three further relations of interest. Let \( A \) be a nice valued submodule of the module \( B \). Then \( G_A(n + 1) = 0 \), so the exact sequence

\[ 0 = F_B(A(n)) \rightarrow F_B(A(n)) \rightarrow G_A(n + 1) = 0 \]

gives

\[ F_B / A(n) \rightarrow F_B(n) \oplus G_A(n + 1). \]

Taking the direct sum with \( F_B(n) \), and using the previous isomorphism, we have

\[ F_B(n) / F_B(n) \rightarrow F_B(n) \oplus F_B(n) \oplus G_A(n + 1). \]

Finally, if \( C \) is a submodule of \( B \), then composing the isomorphism

\[ F_C(n) \circ F_C(n) \rightarrow F_C(n) \text{ and } F_C(n) \rightarrow F_C(n) \rightarrow F_C(n) \rightarrow F_C(n) \circ F_C(n) \circ G(n) \]

provides an isomorphism \( F_B(n) / F_C(n) \rightarrow F_B(n) / F_C(n) \oplus G(n) \). It is readily checked that the latter isomorphism takes \( F_B(n) \) to \( F_B(n) \), whence

\[ F_C(n) \circ F_C(n) \rightarrow F_C(n) \circ F_C(n) \circ G(n). \]

Our exclusion of \( n = 0 \) this point has been to simplify discussion, and is not rooted in any fundamental difference. Indeed, if we define \( F_A(n) \) and \( G_A(n) \) to be the kernel and cokernel, respectively, of the natural map \( A(n) \rightarrow A(n) \), along with the obvious modifcations to the proofs, all the results of this section can be seen to hold when \( n = 0 \). Of course, the convention \( n = 0 \) must be observed. In particular, the fundamental sequence becomes

\[ 0 = F_B(n) \rightarrow F_B(n) \rightarrow F_B(n) \rightarrow G_B(n) \rightarrow Q_B(n) \rightarrow Q_B(n) = 0. \]

To avoid numerous trivial special cases involving these and other invariants, we assume from this point on that all valued modules are reduced.

4. Modules with composition series. A valued module \( M \) is a (nice) composition series if it admits a well-ordered ascending chain of nice submodules \( V_0 \) such that:

1) \( V_0 = 0 \);
2) \( V_n = A \);
3) \( N_0 \oplus b \mid q \); 
4) \( N_0 \in \Gamma \) if \( q \) is a limit.

If \( A \) has a composition series then it is clear that \( A \) is torsion. An every countable valued torsion module has a composition series, we can replace condition 3) by \( N_0 \oplus b \mid N_0 \oplus \bigoplus \mathbb{Q}_p \) and the requirement that \( A \) be torsion.

Modules with composition series can be characterized in terms of their homological dimension in \( \mathbb{Q}_p \).

**Theorem 1.** (Kiselman and Walker [?]). A nonzero valued torsion module has homological dimension one in \( \mathbb{Q}_p \) if and only if it has a composition series.

A module has a composition series if and only if it is simply presented and torsion [2, p5]; we shall be using these two characterizations interchangeably.

Perhaps the most important result concerning modules with composition series was proved by Hill in an unpublished paper [5], and in the following generalized form by Walker [14].

**Theorem 4.** Let \( A, B \) be modules and \( G, H \) nice valued submodules of \( A \) and \( B \), respectively, such that \( A/G \) and \( B/H \) have composition series. Then any \( (\mathbb{Q}_p) \) isomorphism \( G 
\rightarrow H \) extends to an isomorphism \( A \n A \) if and only if \( f_G = f_H \).

Now that \( A \) and \( B \) are not required to be torsion. If \( G = H = 0 \) then \( A \) and \( B \) are modules with composition series and the theorem asserts that \( A \) is isomorphic to \( B \) if and only if \( f_A = f_B \). The latter result was also proved independently by Crawley and Halas [1]. The argument of Fuchs [7, p64] yields:

**Corollary 5.** Let \( A \n A \) be a nice valued embedding with \( B \) a module, and \( f : A \n C \) a homomorphism. If \( B/A \) has a composition series then \( f \) can be extended to a homomorphism \( f : D \n C \).
Since a module \( A \) with a composition series is determined (up to isomorphism) by \( f_A \), it is natural to seek a complement to Theorem 4 which describes those functions which can arise as \( f_A \). This was accomplished by Crawley and Niles. Before stating their theorem, we need some definitions. Let \( f \) and \( g \) be two functions from ordinals to ordinals which vanish beyond some ordinal. We say that \( g \) **dominates** \( f \) if
\[
\sum_{\alpha \leq \beta} f(\alpha) \leq \sum_{\beta \in S} g(\beta)
\]
for all ordinals \( S \). A function which dominates itself is said to be **admissible**.

**Theorem 6.** (Crawley and Niles [1]). If \( A \) has a composition series, then \( f_A \) is admissible. Conversely, if \( f \) is admissible, then there is a module \( A \) with a composition series such that \( f_A = f \).

We refer to the version of Theorem 4 in which \( G = \mathbb{N} = 0 \) as the isomorphism theorem for modules with composition series, and to Theorem 6 as the existence theorem for such modules. In what follows, both of these theorems will be extended to Warfield modules.

5. More invariants. It is clear from the definition that all invariants are properties of the torsion submodule of a module. In this section we define a complementary invariant which is dependent on the elements of infinite order. This invariant was first introduced by Warfield [15] for a restricted class of modules, and later generalised by Stanton [11] to include all modules. We make a further extension to valued modules.

Let \( A \) be a valued module, \( \mathbb{V} = a_0, a_1, \ldots \) a value sequence. Let \( V(\mathbb{V}) \) be the valued submodule of \( A \) defined by
\[
A(\mathbb{V}) = \{ a \in A : V(a) \geq 0 \},
\]
and let \( A(\mathbb{V})^* \) be the submodule of \( A(\mathbb{V}) \) generated by those elements \( a \) of \( A(\mathbb{V}) \) such that \( Vp^k a \leq a_k \) for infinitely many \( k \). Multiplication by \( p \) induces
natural maps
\[
\lambda(\alpha) \quad \lambda(p\beta) \quad \lambda(p^2\beta) \quad \ldots
\]
forming a direct system whose limit we denote \( \mathcal{N}_A(\alpha) \). It is not difficult to see [11, Lemma 1.1] that the maps of this system are all one-to-one. When \( A \) is a module, these maps are also isomorphisms — this follows from the fact that, if \( a \in A \) and \( ha = 0 \), then for each \( a \in A \) there is an element \( b \) in \( A \) such that \( hb = b \) and \( pb = a \). Note that \( \mathcal{N}_A(\alpha) \) is a vector space over the \( p \)-element field, we call the dimension of \( \mathcal{N}_A(\alpha) \) the Harfield invariant of \( A \) at \( \alpha \) and denote it \( \nu_A(\alpha) \). If \( A \) and \( B \) are valued modules whose Harfield invariants are the same for all value sequencers, we write \( \nu_A = \nu_B \). That these invariants provide a measure of the torsion free structure is evident from:

**Lemma 1.** Let \( A \) be a valued submodule of \( B \). If \( B/A \) is torsion then \( \nu_A = \nu_B \). In particular, \( \nu_A = 0 \) if \( A \) is torsion.

**Proof.** If \( b \in B(p^n\alpha) \) then \( p^n b \in A \) for some \( n \), so \( p^n b \in A(p^n\alpha) \) represents an element of \( \mathcal{N}_A(\alpha) \) if and only if \( b \) represents an element of \( \mathcal{N}_A(\alpha) \). The lemma now follows easily.

We list three easily verified properties of Harfield invariants.

(a) If \( A = A_1 \oplus A_2 \) is a valued direct sum, then \( \nu_A(\alpha) = \nu_{A_1}(\alpha) + \nu_{A_2}(\alpha) \).

(b) If \( A \) is torsion free cyclic then \( \nu_A(\alpha) = 1 \) or \( 0 \). Further, if \( A \) is equivalent to \( V(\alpha) \), where \( a \) is a generator of \( A \).

(c) If \( A \) is a (valued) direct sum of torsion free cyclics with basis \( X \), then \( \nu_A(\alpha) \) is the number of elements in \( X \) with value sequencer equivalent to \( \alpha \).

Let \( A \) be a valued submodule of \( B \) with \( B/A \) torsion. If \( A \) is a (valued) direct sum of torsion free cyclics and \( X \) is a basis for \( A \), then \( X \) is called a decomposition basis of \( A \). Modules with decomposition bases play
vital role in the sequel, and one of their most important aspects is their behavior with respect to Harbfield invariants.

Lemma 8. Let $X$ and $Y$ be decomposition bases for the valued modules $A$ and $B$, respectively. Then $M_A = M_B$ if and only if there are substructures $X'$ of $X$ and $Y'$ of $Y$ such that $<X', Y'> = <X, Y>$ (as valued modules).

Proof. Assume that $M_A = M_B$. Using Lemma 7, we have $M_A = M_B = M_B = M_B$ so there is a bijection $\phi: X \rightarrow Y$ such that $V(\phi(x))$ is equivalent to $V(\phi(x))$ for each $x \in X$. Now let $X' = \{x \in X : \phi(x) \neq x\}$ and $Y' = \{y \in Y : \phi(y) \neq y\}$, where $\mu_x$, $\mu_y$ are powers of $p$ chosen so that if $\phi(x) = y$ then $V(\phi(x)) = V(\phi(y))$.

The converse follows directly from Lemma 7.

4. A set theoretic lemma. Before starting the description of Harbfield modules, we prove a set theoretic lemma which will be needed for 'juggling' invariants. Proofs of this lemma for the case $m = N_0$ can be extracted from the proofs of [15, Lemma 21], and [16, Lemma 11]. The case $m = N_0$ is all that is needed in the proof of the isomorphism theorem for Harbfield modules.

Lemma 9. Let $\mathcal{X}$ be a set, $m$ an infinite cardinal, and $F$ a family of subsets of $\mathcal{X}$ such that $\text{card} (F \subseteq \mathcal{X})$ for each $F$ in $F$. Then there exists a function $f: F \rightarrow \mathcal{X}$ such that $f(F) = x$ for all $F \in F$ and such that, for each $x \in \mathcal{X}$, $\text{card} (\{F : f(F) = x\}) = \text{card}(F : x \in F)$ whenever the latter is $\leq m$.

Proof. First, each element $x \in \mathcal{X}$ can be assumed to lie in exactly $m$ members of $F$. Indeed, if $x$ is in fewer than $m$ members of $F$, enlarge $F$ with a set whose only element is $x$. If $x$ is in more than $m$ members of $F$, partition the family of sets $F$ that contain $x$ into subfamilies $S$ of size $m$ and in each $F \in S$, replace $x$ by $x_F$. Of course, this process changes $\mathcal{X}$ and $F$, but after building our function $f$, we revert to the original $\mathcal{X}$ and $F$ in the
7. Warfield modules and the Invariant Theorem. A valued module $A$ is Warfield if there is an exact sequence

$$0 \to C \to A \to B \to 0 \tag{1}$$

in $\mathbb{Q}_p$ with $C$ a direct sum of cyclics and $B$ a valued module with a composition series. We call a sequence of this kind a representing sequence for $A$. Thus a Warfield module is an extension (in $\mathbb{Q}_p$) of a direct sum of cyclics by a simply presented torsion valued module. If $A$ is Warfield and has representing sequences $0 \to C' = A \to A/C' \to B' \to 0$, then we can obtain another representing sequence $0 \to C = A \to A/C' \to C/C' \to B \to 0$ for $A$ by letting $C'$ be the submodule of $C$ generated by any submodule of $A$ basis for $C$. To see this, note that

$$0 \to C/C' \to A/C' \to B' \to 0$$

is exact in $\mathbb{Q}_p$ and $C/C'$ and $B$ have composition series. Hence $A/C'$ has a composition series as required. In particular, we may as well assume $C$ is torsion free and as a basis for $C$ is a nice composition basis for $A$.

We turn now to the isomorphism theorem for Warfield modules.

Theorem 10. (Warfield [15]) Let $A$ and $B$ be Warfield modules. Then $A$ and $B$ are isomorphic if and only if $A \cong B$ and $A^* \cong B^*$. 

Proof. Only sufficiency needs proof. Let $X$ and $Y$ be nice decompositions for $A$ and $B$, respectively, such that $A < X$ and $B < Y$. By Theorem 4, it is enough to choose $X$ and $Y$ so that $A < X$ and $B < Y$. Perhaps surprisingly, we can do this by replacing any first choice of $X$ and $Y$ with suitable subordinates. Examining the equation

$$f_X(x) = g_Y(y) = f_{X,Y}(x)$$

from Section 3, we see that if $f_X(x)$ is infinite, then suitably multiplying the elements of $X$ by powers of $p$ we can replace $X$ by a subordinate such that $f_X(x) = f_{X,Y}(x)$. The trick is to do this for all $x < y$, and this is where our next theorem comes in. Let $a$ be the least of $a$ such that $f_X(x) = f_{X,Y}(x)$, let $a < X$ and let $F = f_{X,Y}(x) = F_a(x + a)$, where $F_a$ consists of those $a$ for which $f_{X,Y}(y) = 0$. If $f$ has the function from $F$ to $X$, then $f$ is $X$-finite. Therefore, by Lemma 5 and if $f(f) > a$, then we multiply $x$ by $p^a$ so that $f_{X,Y}(x) = 0$. Doing the same for $Y$ and $B$, we have

$$0 = f_{X,Y} = A = A/X = 0$$

and

$$0 = f_{X,Y} = B = B/Y = 0$$

exact in $U_p$ with $f_X(x) = f_{X,Y}(x) = f_Y(y) = f_{X,Y}(x)$ whenever $f_X(x)$ is infinite. Note that taking further subordinates of $X$ and $Y$ will not alter these equations. Therefore, Lemma 5 allows us to also arrange that $A < X$.

This ensures that $f_{X,Y}(x) = f_{X,Y}(y)$ for those $x$ for which $f_X(x)$ is finite. Hence $f_{X,Y}(x) = f_{X,Y}(x)$ for all $x$. Finally, the remarks preceding this theorem show that our new $X$ and $Y$ are such that $A < X$ and $B < Y$ have composition series.

Two definitions are needed before we can give the fullest extension (due to Stabell [12]) of Theorem 5 to $A$-valued modules.

Let $A$ be a valued submodule of $B$ and $a$ a value sequence. If $a$ is the natural map $W_A(g) = W_B(g)$, then $a$ is denoted $W_{A,B}(g)$ and its rank is called the $A$-invariant of $B$ relative to $a$. If the
value sequence $\mathbf{e}$. Now let $A$ be a nice valued submodule of $B$. Then $A$ is said to be quasi-sequentially nice in $B$ if, for each coset $b - A$, there is an integer $r$ and an element $a$ in $A$ such that $V(b - a) = V(b - a) + r$.

**Theorem 11** *(Stanton[12]) Let $A, B \subseteq V$ and let $G, H$ be sequentially nice valued submodules of $A$ and $B$, respectively, such that $V/G$ and $V/H$ are Warfield modules. Then any isomorphism $G \rightarrow H$ extends to an isomorphism $A \rightarrow B$ if and only if $f_{A/G} = f_{B/H}$ and $f_{A/H} = f_{B/G}$.

**Proof.** We use the proof of Theorem 10 as a guide, modifying only those arguments that require the additional hypotheses. Again, only sufficient results need be proved.

Let $x \in G : x \in K$ be a nice decomposition basis for $V/G$. Using the fact that $G$ is quasi-sequentially nice in $A$ and taking submodules if necessary, we may assume $V(x) = V(\delta x)$ for all $x \in K$. Then $V\delta x = V(x) = \min(V(x), \delta x) = \max(V(x), \delta x)$ for all $y \in G$. Hence $\delta = \delta y$ and $\delta \cdot y$ are direct in $V$. Set $K = \langle y \rangle$. As $A/K \subseteq G$, the invariance lemma 7 shows that the vector space $V(y)$ is the zero, so $V_{A/K}(y) = \text{Coker}(V_{A/K}(y)) = V_{B/H}(y) = V_{B/H}(y)$. It was observed in Section 3 that

$$f_{A/K}(y) = f_{B/H}(y) = f_{A/G} \circ f_{A/K} \circ f_{A/K}(y).$$

Arguing with this equation and Lemma 8 as we did in the proof of Theorem 10, we can ensure that $f_{A/K}(y) = f_{B/H}(y)$ whenever $f_{A/K}(y)$ is infinite. Thus there is also a corresponding direct sum of cyclics $L \subseteq G$. We may further arrange that $K = L$ and hence that $f_{A/G} = f_{B/H}$ for all $y$. Now $G \otimes K = H \otimes L$, and we are once again in the situation of Theorem 1.

Of course, setting $G = H = 0$ in the preceding theorem gives Theorem 10.

A. The category $C$. We interrupt the study of Warfield modules momentarily to introduce a useful category. Let $C$ denote the category of submodules of the valued module $B$. Define $C$ to be the category whose objects are the objects of $V$ and
where morphism sets are

\[ \text{Hom}_C(A, B) \cong \text{Hom}_p(A, B) / \text{Hom}_C(A, B). \]

Note that \( C \) is additive with kernels and arbitrary infinite direct sums; the kernel of \( f : \text{Hom}_C(A, B) \to \text{Hom}_C(A, C) \) is \( f^{-1}(0) \), while the direct sum in \( C \) of the family \( \{ B_i \} \) is just the direct sum in \( V_p \). The next theorem characterizes \( C \)-isomorphism in terms of the category \( V_p \).

**Theorem.** Let \( A, B \in C \). Then \( A \) is isomorphic to \( B \) in \( C \) if and only if there are coproducts \( S \) and \( T \) in \( V_p \) such that \( A \otimes S \) and \( B \otimes T \) are isomorphic as \( V_p \)-valued modules.

**Proof.** Suppose \( f : A \otimes S \to B \otimes T \) is an isomorphism in \( V_p \). Let \( \lambda \) be the injection of \( A \) into \( A \otimes S \). \( \psi \) the projection of \( A \otimes S \) onto \( B \); we denote the corresponding injections and projections associated with \( S, B \) and \( T \) in a similar fashion. We claim that \( \psi f \lambda = \text{Hom}_C(A, B) \) is a \( C \)-isomorphism with inverse \( \psi f^{-1} \lambda = \text{Hom}_C(A, B) \). Now

\[
\begin{align*}
\lambda &= \psi f^{-1} \lambda \cdot \text{Hom}_C(A, B) \\
\lambda &= \psi f^{-1} \lambda \cdot \text{Hom}_C(A, B) \\
\lambda &= \psi f^{-1} \lambda \cdot \text{Hom}_C(A, B) \\
\lambda &= \psi f^{-1} \lambda \cdot \text{Hom}_C(A, B)
\end{align*}
\]

while \( \psi f^{-1} \lambda = \psi f^{-1} \lambda = \text{Hom}_C(A, B) \). Similarly for \( \psi \).

Conversely, if \( A \) is isomorphic to \( B \) in \( C \), there are valued maps \( f : A \to B \) and \( g : B \to A \) such that

\[ gf \cdot \text{Hom}_C(A, B) \cdot 1_A = 1_A \cdot \text{Hom}_C(A, B) \]

and

\[ fg \cdot \text{Hom}_C(B, C) \cdot 1_B = 1_B \cdot \text{Hom}_C(B, C) \]

Thus there is \( a \in \text{Hom}_C(A, B) \) and \( b \in \text{Hom}_C(A, B) \) with \( gf = a \cdot 1_A \) and \( fg = a \cdot 1_B \); we complete the proof by showing that \( A \otimes S \) is isomorphic to \( B \otimes S \). First observe that composing the last mentioned equations with \( f \) yields \( fgf = a \cdot f \) and \( fgf = a \cdot f \), respectively, and hence \( f = a \cdot f \) and \( g = a \cdot g \). Using these, it is readily checked that the required isomorphism is given by \( (\alpha, \beta) \mapsto (g(a) \circ \alpha, a(\beta) \circ g(a)) \) with inverse given by
We have the following isomorphic refinement theorem in $C$.

**Theorem 13.** If $A$ is a summand (in $F$) of a direct sum of rank one modules $M_i : i \in I$, then $A$ is $C$-isomorphic to the direct sum of modules $N_j : j \in J$ for some $J \subseteq I$.

**Proof.** Since $C$ satisfies the conditions of [13, Theorem 5], it suffices to prove that the endomorphism ring of a rank one module is local. Let $A$ have rank one, $0 \neq a$ an element of infinite order in $B$, and $F \neq \text{End}_C(B)$ a nonzero $C$-endomorphism of $B$. If $a \neq 0$ then there are positive integers $m, n$ such that $p^m(a) = u p^n$ for some unit $u$. Obviously $x \neq 0$ and $p^m x$ is unique, so the assignment $x \mapsto p^m x$ defines an isomorphism of the $C$-endomorphism ring of $B$ with $\mathbb{Z}_p$. Similarly, when $y \neq 0$ it can be shown that the $C$-endomorphism ring of $B$ is isomorphic to $C$.

Our next result is an immediate consequence of the preceding two theorems.

**Corollary 14.** Let $A \in V_p$ be a summand of a direct sum of modules of rank one.

Then there is a torsion module $T$ such that $A \cong T$ is a direct sum of modules of rank one.

**8. Summands of Peculiar Modules are Prismatic.** In order to prove this assertion, we first explore a little of the relationship between prismatic modules and simply presented modules. Let $A = S(y)$ be simply presented and let $x = V_1 S(y)$ be the decomposition given in Lemma 1. In each $V_i$ choose an element $x_i$ so that, if possible, $x_i$ has infinite order. If $x_1$ is the valued subset of $V$ consisting of the $x_i$'s, all their $p$-multiples, and 0, then
0 \cdot \mathcal{D}(X) \cdot \mathcal{D}(Y) \cdot \mathcal{D}(Y/X) = 0

is a representing sequence for \( \mathcal{D}(Y) \) as a Warfield module. Thus \( X \) is Warfield.

However, Warfield modules need not be simply presented - the exact relationship between these two classes of modules will be described in Section 10; for the present, we show that with the 'addition' of a suitable torsion module, a Warfield module becomes simply presented.

**Lemma 15.** Let \( A \) be a Warfield module. Then there is a simply presented torsion module \( B \) such that \( A \ast B \) is simply presented.

**Proof.** Let \( 0 \to C \to A \to A/C \to 0 \) be a representing sequence for \( A \). Then \( T(C) \) is simply presented (recall that the functor \( T \) described in Theorem 2 takes a valued module to a module). By Theorem 6, there is a simply presented torsion module \( B \) such that \( f_0 \circ f_1(C) \mathcal{M}_0 \) and \( f_1 \circ f_0 \mathcal{M}_0 \), for \( T(C) \otimes B \) and \( A \ast B \) have the same Ulm and Warfield invariants, so Theorem 10 implies \( T(C) \otimes B = A \otimes B \).

There are in fact a number of different ways to prove Lemma 15. We present one other, based on the following lemma which is also needed for later work.

**Lemma 16.** Let \( A \) and \( B \) be modules, \( f : A \to B \) and \( g : B \to A \) homomorphisms, and \( C \) a submodule of \( A \) such that \( g \circ f \) is the identity on \( C \). Then \( (D(C) \otimes B = A \otimes (Bf(C)) \). The isomorphism is given by

\[ (a + C, b) \mapsto (a - g(b) - f(a), b - f(a) + f(C)), \]

with inverse given by

\[ (a, b + f(C)) \mapsto (a - g(b) + C, b + f(a) - g(C)). \]

The proof is trivial.

**Second proof of Lemma 15.** Let \( 0 \to C \to A \to A/C \to 0 \) be a representing sequence for \( A \). As we observed in Corollary 5, there are maps \( f : A \to T(C) \) and \( g : T(C) \to A \) which are the identity on \( C \). Lemma 16 yields \( (A/C) \otimes T(C) = \)}
Theorem 17. A summand of a Warfield module is Warfield.

Proof. Let $A \oplus B \oplus D$ be Warfield and suppose $B \neq 0$. By Lemma 15, we may assume $A$ is simply presented. Thus $A = A_1$ where each $A_j$ is simply presented of this kind. Theorem 15 implies $A_2$ is $C$-isomorphic to $A_1 \oplus H_j$ for some $j \leq I$, so there are maps $f : A \to A_2$, $g : B \to A_2$, $h : A_2 \to H_j$ with $g f + h = 1_A$ and $f g + h = 1_B$. Now choose a representing sequence $0 = C \to A \to A/C \to 0$ for $A$. As $A$ is torsion, we can arrange that $C$ is chosen so that $g(C) = 0$ and hence $g f$ is the identity on $C$. We show that $0 = C \to A \to A/C \to 0$ is a representing sequence for $A$. Lemma 15 gives an isomorphism

$$h : (A/C) \oplus B \oplus (B/f(C))$$

and hence an isomorphism

$$h' : (A/C) \oplus B \oplus (B/f(C)) \to D$$

Using the fact that $ImB$ is torsion and the description of $h$ given in Lemma 15, it is readily checked that $B/f(C)$ is torsion. Thus it is possible to choose $E \leq B \oplus D$ so that $A/E$ is simply presented and $h'(E) = A \oplus D$. But then

$$B = B/f(C) \to f(C) \oplus (B/f(C)) \to D$$

with the left hand side simply presented and torsion, and therefore $B/f(C)$ is simply presented and torsion. Obviously $f(C)$ is a direct sum of cyclics, and it remains to show that $f(C)$ is nice in $B$. If $c \in C$ then $h^{-1}(c, 0) = (c, f(c))$. Thus the isomorphism $h^{-1}$ sends $C \subset A$ to $f(C) \subset D$ so $f(C)$ is indeed nice in $B$.

Corollary 18. A module is Warfield if and only if it is a summand of a simply presented module.

Proof. We have already seen that if $A$ is Warfield then $A$ is a summand of a
simply presented module (Lemma 15). Conversely, if \( A \) is a summand of a simply presented module, then \( A \) is a summand of a Warfield module and Theorem 17 implies \( A \) is Warfield.

10. Existence theorems. Necessary and sufficient conditions for the existence of a Warfield module with prescribed Ulm and Warfield invariants were given in [4]. Since [4] is written from the valued viewpoint and the relevant notation and definitions are those we are presently employing, we report the results of [4] without proofs.

First, the main existence theorem.

**Theorem 19.** Let \( C \) be a direct sum of infinite cyclic valued modules and \( f \) a function from the ordinals to the cardinals that vanishes beyond some ordinal. Then there exists a Warfield module \( A \) with \( f_A = f \) and \( \nu_k = \nu_c \) if and only if \( f \) dominates \( f = f_C \) and \( f \neq f_C \).

Note that if \( C \) is a direct sum of cyclic of the kind mentioned in the theorem, then \( \hat{\mathcal{G}}_{\text{ord}}(C) \) is determined, to within a finite cardinal, by the Warfield invariants of \( C \). Thus Theorem 19 does indeed give conditions for the existence of a Warfield module with prescribed Ulm and Warfield invariants. The proof in [4] makes extensive use of the derived Ulm invariants and is in fact constructs a representing sequence

\[ D = C = A + A/C = C. \]

for \( A \). Of course, Theorem 19 generalizes Theorem 6.

It was stated in Section 9 that a Warfield module need not be simply presented. The reason for this is a lack of relative Ulm invariants in sense made precise by:

**Theorem 20.** ([4]) Let \( A \) be a Warfield module and \( X \) a nice decomposition basis of \( A \) with \( A/\langle X \rangle \) simply presented. Then \( X \) is simply presented with spanning tree containing \( X \) if and only if \( f_{A/\langle X \rangle} \) dominates \( f_{A,<X>} \) for all \( f_{A,<X>} \).
The first example of a Warfield module which is not simply presented was given by Brown and Yau [10]. Their example has rank two. The following rank one example is due to Warfield [22].

Example 11. Let $A$ be the Warfield module with decomposition basis $\mathfrak{x}$ such that $V(x) = 0, 2, 4, \ldots$ and $f_{A, C, X}(x) = 1$ if $x = w$ and 0 otherwise. Such a module exists (Theorem 12), while Theorem 20 shows that $x$ cannot be extended to a spanning tree. Since every element of infinite order in $A$ has valence sequence equivalent to $V(x)$, it follows that $A$ is not simply presented.

Theorem 19 can be used to construct a Warfield module of countably infinite rank which is not simply presented.

Example 12. Let $C = C_d$ where $C_d$ is generated by $x$ such that

$$w_0^0 x_n = n + n.$$ 

By Theorem 19, $C$ can be embedded nicely in a Warfield module $A$ with the same 0-invariants as $C$, such that $A/C$ is torsion and simply presented (actually, we are using the fact that, in the proof of Theorem 6, $A$ is constructed with representing sequence $0 = C - A = A/C + 0$). We claim that each homomorphism $A$ is either finite or of finite index. To see this, let $A = B \oplus D$ and let $x = 0, 2, 4, \ldots$. Since $w_1(x) = 0$ we may assume that $w_1(x) = 0$. Then $x$ must contain an element $y$ such that $w_0^0 y = 2m + n$ for some $m$ and all $n$. Hence $f_2(x) + n = 0$ for all $n$, so $f_2(x) = 0$ for all $k = 2m$. This implies that the torsion submodule of $D$ is bounded, and to $D$ is bounded since $A$ admits to nonzero torsion-free submodules.

In the light of these examples, one might ask whether there are Warfield modules of arbitrary rank which are not simply presented and which do not decompose into modules of smaller rank. The answer is no, and the next theorem shows that all such examples must be of countable rank.
Theorem 23. ([4]) If A is Warfield, then $A = B \oplus C$ where B is simply presented and there is an ordinal $\alpha$ such that $\beta^\alpha C$ is countable and $C/\beta^\alpha C$ is torsion. In particular, C has countable rank.

We saw in Section 9 that a Warfield module can be made simply presented by *adding* a suitable torsion module. The same result is achieved by taking the direct sum of enough copies of a Warfield module:

Theorem 24. ([4]) If A is a Warfield module, then the direct sum of infinitely many copies of A is simply presented.

Let $X$ be a decomposition basis of A. We say that $X$ is a splitting decomposition basis of A and A splits over $X$ if A can be written as the direct sum $A = X \oplus A_X$ with $X \oplus A_X$. Thus a module with a splitting decomposition basis is a direct sum of rank one modules. Decomposition bases which generate isomorphic direct sums of ranked modules are said to be isomorphic. We give an example to show that isomorphic decomposition bases in a module need not have the same splitting properties.

Example 25. Let $G$ be the countably infinite direct sum of copies of the module A in Example 22. Then the decomposition basis $X$ of G formed by taking the decomposition basis described in Example 22 from each copy of A is not splitting (just check the relative Ulm invariants). However, the module $H = A + A_X$ is simply presented and has a splitting decomposition basis isomorphic to the decomposition basis of A (Theorem 20). Hence the countably infinite direct sum of copies of H has a splitting decomposition basis isomorphic to X. But G and H have the same Ulm and Warfield invariants and so are isomorphic.

We now talk about characterizing those isomorphism classes of decomposition bases in a Warfield module which contain a splitting decomposition basis. Some preparatory definitions are needed.
Let \( f \) and \( g \) be two functions from ordinals to cardinals which vanish beyond some ordinal. If \( f \leq g \), we define a function \( f' \) by:

\[
f'(\alpha) = \begin{cases} 
  f(\alpha) & \text{if } f(\alpha) \text{ is finite} \\
  g(\alpha) & \text{otherwise}.
\end{cases}
\]

Obviously \( f' \leq g \). Given an ordinal \( \lambda \), we say that \( f \) is \( \lambda \)-admissible if the function \( f' \) defined by

\[
f'(\alpha) = \begin{cases} 
  f(\alpha) & \text{if } \alpha < \lambda \\
  0 & \text{otherwise}
\end{cases}
\]

is admissible. The length \( \text{length}(f) \) is defined as \( \sup \{ \alpha : f(\alpha) > 0 \} \), we say that \( f \) has length \( \lambda \) if \( \lambda \) is \( f \)-admissible and \( \alpha \in f(\alpha) \) for all \( \alpha < \lambda \).

**Lemma 26.** (4, Lemma 21) Let \( f \) and \( g \) be functions from ordinals to cardinals which vanish beyond some ordinal. If \( f \) dominates \( f' \cdot g \), and \( g = \frac{f'}{f} \cdot g \), then \( f \) can be written \( f = \sum_{\alpha < \lambda} f_{\alpha} \cdot g_{\alpha} \) such that \( f_{\alpha} \) dominates \( f'_{\alpha} \cdot g_{\alpha} \) for all \( \alpha < \lambda \).

**Theorem 2.** Let \( A \) be a von Neumann module with decomposition basis \( B \). Then \( A \) is a splitting decomposition basis isomorphic to \( X \) if and only if \( \text{rank}(A) = \text{rank}(X) \) and \( A \) has length at least \( \text{length}(X) \).

**Proof.** Let \( \text{rank}(A) = \text{rank}(X) \), and \( \lambda = \text{length}(X) \). Suppose \( A \) has a splitting decomposition basis isomorphic to \( X \). Since \( \text{rank}(A) \) depends on \( X \) only up to isomorphism, we may assume that \( X \) itself is splitting. First assume that \( \lambda \) contains but one element \( x \). For each \( a < \lambda \), there are only finitely many elements \( e \in X \) such that \( a < b + a \) and so \( \text{rank}(A) \) dominates finitely to \( f_{\alpha} \) in this range. Since \( \text{rank}(A) \) is \( \lambda \)-admissible, \( A(\alpha) = \lambda_\alpha \). It follows that \( A \) has length at least \( \lambda \) and is \( \lambda \)-admissible. Now let \( X \) be arbitrary and suppose \( a < \lambda \). If \( \sum_{\alpha < a} f_{\alpha}(A) = \sum_{\alpha < a} f_{\alpha}(X) = \sum_{\alpha < a} f_{\alpha}(x) \) and there is nothing to prove. If \( \sum_{\alpha < a} f_{\alpha}(A) = \sum_{\alpha < a} f_{\alpha}(x) \), then, since there is an element \( x \) in \( X \) with \( a < \text{length}(X) \) and a rank one module \( A \) containing \( a \), the first part of
the argument shows

\[ \sum_{\gamma < \alpha} \mathcal{R}^*(\alpha) \geq \sum_{\gamma < \alpha} (f_\alpha - f_\gamma)(\alpha) \approx \chi_\alpha \sum_{\gamma < \alpha} f_\alpha(\gamma) \approx \sum_{\gamma < \alpha} f_\alpha(\gamma) = f_\alpha(\alpha). \]

This also proves that \( f^* \) has length at least 1.

Conversely, suppose \( f^* \) satisfies the conditions of the theorem. We claim that \( f^* \) dominates \( f^* \cdot \mathcal{R}_{\alpha^*} \). Since \( f_{\alpha^*} \) dominates \( f_{\alpha^*} \cdot \mathcal{R}_{\alpha^*} \) (Theorem 19), it suffices to consider only those \( \alpha < \lambda \) for which \( \sum_{\gamma < \alpha} f_\alpha(\gamma) \leq \chi_\alpha \) but then the condition on the length of \( f^* \) ensures that

\[ \sum_{\gamma < \alpha} f_\alpha(\gamma) \leq \chi_\alpha \sum_{\gamma < \alpha} f_\alpha(\gamma) \leq \sum_{\gamma < \alpha} f_\alpha(\gamma) = f_\alpha(\alpha). \]

and the claim is established. Now we use Lemma 26 to write \( f^* = \sum f_\alpha \) where \( f_\alpha \) dominates \( f_\alpha \cdot \mathcal{R}_{\alpha^*} \) for each \( \alpha \in \lambda \). But then \( f^* = \sum f_\alpha \) dominates \( f_\alpha \cdot \mathcal{R}_{\alpha^*} \) for all \( \alpha \in \lambda \), so there is a Warfield module \( A_\lambda \) with representing sequence \( 0 \to X = A_\lambda - A/\text{id} \to 0 \) such that \( f_{\alpha^*} = f_\alpha + \mathcal{R}_{\alpha^*} \). Now \( A_\lambda \) has the same Ulm and Warfield invariants as \( A \), and so is isomorphic to \( A \).

Theorem 27 is useful in determining when a Warfield module is a direct sum of rank one modules. For example, the next theorem is a generalization of a result of Notman [2].

Corollary 28. Let \( A \) be a Warfield module with decomposition basis \( X \). If there is an ordinal \( \alpha \) such that \( a = x \alpha + a + 1 \) for all \( x \in X \) then \( A \) is a direct sum of rank one modules.

Proof. This result follows from Theorem 27 by observing that \( \mathcal{R}(A/\text{id}) \) and \( f_\alpha(\alpha) = \chi_\alpha(\alpha) = f_\alpha(\alpha) \) for all \( \alpha < \lambda \).

11. Decomposition basis. The notion of a nice decomposition basis is central to our development of Warfield modules. In this section we take a closer look at decomposition bases, with applications to Warfield modules in mind. First, a (semi)module with a decomposition basis has a decomposition basis - this result was first proved by Warfield [11]. Of particular interest is the question of whether a module with a decomposition basis also has a nice decomposition basis.
Since subcoordinates of nice decomposition bases are also nice, the following result of Warfield [15 Lemma 4], shows that it is sufficient to look for nice decomposition bases among the subcoordinates of any given decomposition basis.

**Lemma 25.** Let $\lambda$ and $\gamma$ be decomposition bases of a module $G$. Then there exist subcoordinates $\lambda'$ and $\gamma'$ of $\lambda$ and $\gamma$, respectively, such that \[ \langle \lambda' \rangle = \langle \gamma' \rangle . \]

We say that a valued module $A$ has **finite jump type** if, given an ordinal $\alpha$, there are at most finitely many $\alpha < \beta$ such that $\nu x = \alpha$ and $\nu x + 1$ for some $x$ in $A$. It turns out that the condition $\alpha < \beta$ in the preceding definition can be replaced by $\nu y = \alpha$ for $\gamma < \omega$. It is important to notice that finite jump type is a condition on the number of distinct kinds of value sequences occurring in a valued module, rather than on the number of elements with these value sequences. This point is illustrated by the fact that the (valued) direct sum of arbitrarily many copies of the same valued cyclic has finite jump type.

**Lemma 26.** Let $x$ be a decomposition basis of the valued module $A$, and let $y$ be an element of $x$. If $y < x$ has no element of maximum value, then there are finite subsets $E_1 < E_2 < \ldots$ of $x$, and elements $y_j \in \langle E_j \rangle$ such that

1. $y_j = \nu(y - x_j)$ is the maximum value of elements in $y < x_j$.
2. $\nu y_j = \nu y_j < \nu y_{j+1} < \ldots$
3. Any coordinate $x_j$ in $\langle E_j \rangle \setminus \langle E_{j+1} \rangle$ has value $a_j$.

**Proof.** Let $x_0 = 0$ and $E_0 = \emptyset$, and suppose the construction has been carried out to $j$. Since $y - x_{j+1}$ is not of maximum value in $y < x_j$, there is a $z$ in $x$ such that $v(y - x_{j+1} + z) > v y_j$. We can choose $z$ so that every nonzero coordinate of $z$ has value $a_{j+1}$. Let $E_j = E_{j+1}$ together with the elements of $x$ where $x$ has a nonzero coordinate. Since $E_j$ is finite there is $x_j$ in $\langle E_j \rangle$ maximizing $\nu(y - x_j)$. Then $\nu(x_j - x_{j+1} + z) = a_{j+1}$ so, since $v y_j < \nu x_j$, every coordinate of $x_j$ in $\langle E_j \rangle \setminus \langle E_{j+1} \rangle$ must have value $a_{j+1}$. 
Theorem 32. Let $X$ be a decomposition basis of the valued module $A$. If $X$ has finite jump type, then $X$ is nice in $A$.

Proof. If $X$ is not nice in $A$, then there is $y$ in $A$ such that $y + <x>$ has no element of maximum value. Let $y$, $x_j$, and $a_j$ be as in Lemma 10, and suppose $y + <x>$ has no element of maximum value. For sufficiently large $j$, say $j \geq n$, the coordinates of $p_n^j$ in $S_j \setminus S_{j-1}$ are zero. By Lemma 10, 11, we may choose $a_j \in S_j \setminus S_{j-1}$. Let $a_j$ be the projection of $x_j$ on $<a_j>$. Then $\forall j \geq n$, contradiction the assumption that $<x>$ has finite jump type. Note that $p_n^j x_j$ is the projection of $p_n^j x_j$ on $<a_j>$. If $i > j$, then $v(x_i - x_j) = v(y + x_j - y + x_i) + a_j$ is the projection of $x_i$ on $<a_j>$ and has value $a_{j-i}$. Thus $v^p a_j + v^p a_{i-j} + v^p a_i \leq a_j$ for $i > j \geq n$, whereupon $v^p a_j = \sup a_j$.

In particular, a finite decomposition basis generates a nice submodule. We give an application of this in:

Theorem 32. Let $A$ be a Warfield module of finite rank. If $A$ has a splitting decomposition basis, then $A$ splits over every decomposition basis.

Proof. It is easy to see that if $A$ has a decomposition basis $X$ satisfying the conditions of Theorem 27, then every decomposition basis satisfies those conditions — the main point being that finite changes to a function do not affect its admissibility. Let $X$ be any decomposition basis of $A$. By Theorem 27 there is a splitting decomposition basis $Y$ of $A$ which is isomorphic to $X$. Now $A$ has finite rank so $E_A \cong E_{A_Y}$. While Theorem 21 shows that $<x>$ and $<v>$ are nice in $A$, Theorem 4 extends the isomorphism $<v> \cong <x>$ to an automorphism of $A$. This automorphism writes a splitting of $A$ over $Y$ into a splitting of $A$ over $X$. 
Theorem 35. Every countable decomposition basis of a valued module has a nice subdirect.

Proof. Let \( x_1, x_2, \ldots \) be a decomposition basis and \( a_1, a_2, \ldots \) an enumeration of the valuations \( v(p^n x_i) \). Choose nonnegative integers \( n(j) \) such that, for each \( i \neq j \), either
\[
v_{x_i} < v(p^{n(j)} x_j),
\]
or
\[
v_{x_i} = v(p^{n(j)} x_j) \quad \text{for all } n.
\]
Then \( v(p^{n(j)} x_j) : j = 1, 2, \ldots \) has finite jump type and hence is nice.

Thus if \( A \) and \( B \) are direct sums of countable modules and both \( A \) and \( B \) have decomposition bases, then \( A \) and \( B \) have nice decomposition bases. It follows that \( A \) and \( B \) are isomorphic to \( B \) if and only if \( t_A = t_B \) and \( v_A = v_B \). This result was first proved by Herstein[15, Theorem 3]. We now show that there is a module which has a decomposition basis, but which has no nice decomposition basis.

Lemma 34. Let \( A \) be a valued module with decomposition basis \( X \). Let \( \{x_j\} \) be a sequence in \( X \) such that \( v(x_j) < v(x_{j+1}) \) and \( v(x_j) \leq \sup v(x_k) \) for all \( j \). Then there is a valued module \( B \) containing \( A \) as a subvalued submodule, and an element \( y \) in \( B \), such that \( i(y) \) is torsion and
\[
v(y + x_1 + \cdots + x_n) = v(x_n)
\]
for \( n = 0, 1, 2, \ldots \).

Proof. First observe that it suffices to verify the inequalities
\[
v(y + x_1 + \cdots + x_n) \leq v(x_n),
\]
for if \( v(y + x_1 + \cdots + x_n) > v(x_n) \), then
\[	v(y + x_1 + \cdots + x_n) > v(x_n + y).
\]
If such a \( y \) exists in \( A \), then set
\( B = A \). Otherwise construct \( B \) as the group direct sum of \( A \) and a cyclic group of order \( p \) generated by \( y \), valued as follows:
\[
B(a) = A(a) \quad \text{for } a = v(x_n);
\]
\[
B(a) = A(a) + y + x_1 + \cdots + x_n \quad \text{for } a \leq v(x_n + y),
\]
\( n = 0, 1, 2, \ldots \).
note that $B(x) = A(y)$ so we will be done if we show that the $B(\alpha)$ define a valuation on $B$. Clearly the $B(\alpha)$ are decreasing in $\alpha$, and $B(\alpha) : B(\alpha+1) \subseteq B(\alpha+1)$. Suppose $\alpha$ is a limit ordinal. If $x \in \text{sup} \, V(\alpha)$, then
\[ \forall y(\beta) = \forall (\alpha) = A(\alpha) = B(\alpha). \]

If $x \in \forall \,(x_{\alpha+1})$, then
\[ \forall (\beta) = \forall (\alpha) = A(\alpha) = B(\alpha). \]

But every $x_i x_j \cdots x_i = \forall (\beta) = A(\alpha) = B(\alpha)$, so
\[ \forall (\beta) = A(\beta) = \forall (\alpha) = B(\alpha). \]

Finally, suppose $\beta \in \text{sup} \, V(\alpha)$ and that $x \in B(\beta)$ for some $z$. If $x \in B(\beta)$, we can write $x = y x_i x_j \cdots x_k$ for some $y \in A(\beta)$. Therefore
\[ \forall (\beta) = A(\beta) = B(\beta) = A(\alpha) = B(\alpha). \]

Corollary 39. Let $A$ be a valued module. Then there is a module $B$ containing $A$ as a valued submodule such that $B/A$ is torsion and, whenever $x_j$ is a sequence in a decomposition basis for $B$ such that $V(x_j) < V(x_j)$ and $V(x_j) \geq \forall V(x_j)$ for all $j$, there is a $y$ in $B$ such that
\[ \forall (y, x_1, \ldots, x_k) = V(x_j). \]

for $n = 1, 2, \ldots$.

Proof. Transfinite application of Lemma 38 gives a valued module $B'$ with the desired properties. Now use Theorem 2 to put $B'$ in a module $B$ such that $B/B'$ is torsion.

Theorem 36. If $A$ is a direct sum of infinite cyclic valued modules that does not have infinite jump type, then a basis for $A$ is a not nice decomposition basis for some module $B$.

Proof. Let $A$ be as in Corollary 39. Since $A$ does not have infinite jump type,
we can find a sequence \( \{x_j\} \) in some subordinate decomposition basis \( A \) such that \( v(x_j) < v(x_{j+1}) \), and \( v(x_j) > v(y) \) for all \( j \). Let \( y \) be as in Corollary 25. Then \( y + A \) has no element of maximal value, for \( v(y + z) = v(y) + z \) for all \( z \), which is impossible.

**Theorem 31**: There is a module \( B \) with a decomposition basis \( X \) such that no subordinate decomposition basis is nice in \( B \).

**Proof**: Let \( S \) be the set of all sequences \( s \) of ordinals \( x_0 = 0 < x_1 < \ldots < x_n \) where \( v(x_j) < j \). For \( s \in S \) let \( x_s \) generate a cyclic valued module \( X_s \) such that \( v(x_s) = x_n \). Let \( A \) be the direct sum of the \( X_s \), let \( X = \{x_s : s \in S\} \), and let \( B = X / A \) as in Corollary 25. Let \( f(s) \) be a nonnegative integer for each \( s \in S \), and let \( X' = \{f(x_s) : s \in S\} \). We shall show that \( \langle X' \rangle \) is not nice in \( B \).

The set \( \{s : f(s) \neq 0\} \) is infinite for some \( n \) least there be an \( s \in S \) such that \( x_0 > x_n \) whenever \( f(x) = n \). There exist \( x_s \) in \( X' \) with \( v(x_s) < v(x_2) < \ldots < v(x_{n+1}) = \sup v(x_s) \). Corollary 25 and the last line in the proof of Theorem 30 show that \( \langle x_s \rangle \) is not nice in \( B \).

It follows from Lemma 25 that a module of the kind described in Theorem 31 has no nice decomposition basis.

12. **Projective characterization and balanced projective modules**. Recall that a sequence \( 0 \to A \to B \to C \to 0 \) of valued modules is in List \( (C, A) \) if and only if, for all ordinals \( \alpha \),

\[
0 \to A(\alpha) \to B(\alpha) \to C(\alpha) \to 0
\]

is exact if \( \alpha \) is a sequence of modules, and a valued module is projective if and only if it is free. A sequence \( 0 \to A \to B \to C \to 0 \) is called sequentially pure if the
Sequence of modules:

\[ 0 \rightarrow A(x) \rightarrow B(y) \rightarrow C(z) \rightarrow 0 \]

is exact for all valued sequences \( z \). It is readily seen that a valued module is projective with respect to all sequentially pure exact sequences if and only if it is a direct sum of cyclics.

When attention is restricted to such sequences and their relative projectives within the category of modules, we have:

**Theorem 39. (Warfield [19]) (A) A module is projective relative to all short exact sequences of modules which are in \( \tau \), if and only if it is Warfield and contains a free decomposition basis.

(B) A module is projective relative to all sequentially pure sequences of modules if and only if it is Warfield.

The modules described in part (A) of the preceding theorem are called balanced projectives. They comprise a particularly tractable class of Warfield modules which we now examine in some detail. Let \( A \) be balanced projective. Then \( A \) contains a free valued subgroup \( C \) such that \( A/C \) has a composition series. It is immediate from Theorem 11 that \( C \) is nice in \( A \), and hence a balanced projective module is Warfield. Thus a module \( A \) is balanced projective if and only if \( A \) has a representing sequence

\[ 0 \rightarrow C 
\begin{array}{c}
\rightarrow A \\

\rightarrow A/C \rightarrow 0
\end{array}
\]

with \( \text{hom.dim.} C = 2 \) and \( \text{hom.dim.} A/C \leq 1 \). Hence \( \text{hom.dim.} A \leq 1 \) for any balanced projective module \( A \).

**Lemma 39.** Let \( A \) be a balanced projective module and \( C \) a free valued subgroup of \( A \). Then \( A/C \) has a composition series.

**Proof.** By Theorem 32, \( A/C \rightarrow A 
\begin{array}{c}
\rightarrow A/C \rightarrow 0
\end{array}
\]

is a \( \tau \)-extension. As \( \text{hom.dim.} C = 0 \) and \( \text{hom.dim.} A \leq 1 \), it follows that \( \text{hom.dim.} A/C \leq 1 \).
Balanced projective modules are in fact simply presented.

**Theorem 40.** Let $A$ be a balanced projective module with free decomposition basis $x$. Then $A = \sum A_k$ where each $A_k$ is rank one simple presented and contains $x$.

**Proof.** By Theorem 19, $f_A$ dominates $f_A \cdot f_{x'}$, but $f_{x'} = 0$ so $f_A = f_A \cdot f_{x'}$. Then Theorem 20 completes the proof.

The Warfield invariants of a balanced projective module are particularly easy to describe. Let $a = a_0, a_1, ...$ be a value sequence. If $a$ is a selected projective module and $v_A(a) = 0$, then there is a positive integer $i$ such that $a_{i+k} = a_{i+k}$ for $k = 0, 1, ...$. Thus we might as well consider only those sequences $a, a + 1, a + 2, ...$ where $a$ is a limit ordinal, and hence only limit ordinals. With this in mind, let

$$n(a,n,x,y) = \begin{cases} 1 & \text{for } n \text{ a limit, and} \\ 0 & \text{otherwise.} \end{cases}$$

For a balanced projective module $A$ and limit ordinal $\alpha$, it is clear that $v_A(a)$ is the dimension of the vector space $\mathbb{P}^M/\mathbb{P}^M(a) + (v^M)_A$; that is, $v_A(a)$ and the invariant $v(a, x)$ defined by Warfield [14, 15] agree on balanced projective modules. The isomorphism and existence theorems for balanced projective modules (due to Warfield [17]) now follow directly from Theorems 12 and Theorem 13, respectively.

**Theorem 41.** Let $A$ and $B$ be balanced projective modules. Then $A$ is isomorphic to $B$ if and only if $f_A = f_B$ and $h_A = h_B$.

**Theorem 42.** Let $f$ and $h$ be functions from ordinals to cardinals which vanish beyond some ordinal. Then there is a balanced projective module $A$ with $f_A = f$ and $h_A = h$, if and only if $f$ dominates $f \cdot h$. 
Of course, there is also an analogue of Theorem 11 for balanced projective modules. More interesting is the following result which was proved in [19, Corollary 46] and is essential for the localisation arguments used in that paper.

**Theorem 45.** Let $A$ and $B$ be isomorphism balanced projective modules with free decompostion bases $X$ and $Y$, respectively. Then any isomorphism $X \cong Y$ extends to an isomorphism $A \cong B$.

**Proof.** We have already seen that $A_X$ and $B_Y$ are not submodules of $A$ and $B$, respectively, such that $A/A_X$ and $B/B_Y$ have composition series. Now $f_{A_X} \cdot f_A = f_B \cdot f_{B_Y}$ follows from $f_{A_X} \cdot f_{B_Y} = 0$ and Theorem 4 completes the proof.

**Remark.** The importance of the preceding theorem lies in the way it differs from Theorem 11. The requirement that $A/A_X$ and $B/B_Y$ be simply presented does not appear in the statement of Theorem 45 as its equivalent does in Theorem 11, and this is the main obstruction to a direct application of Theorem 4. This problem was overcome in [19] by passing to modules over a complete discrete valuation ring. However, Lemma 30 is exactly what is needed (see also the remark following the statement of [19,Theorem 4.4]).

15. The torsion submodule of a Warfield module. The torsion submodule of a balanced projective module is called an $S$-module. In this section we show that the torsion submodule of a Warfield module is an $S$-module. Hence, the larger class of Warfield modules does not, in this way at least, lead to a larger class of torsion modules. $S$-modules were first studied by Warfield in [17] and we refer the reader to that paper for a full account.

The torsion submodule of a rank one Warfield module is characterized in:

**Theorem 44 [6].** Let $A$ be a rank one Warfield module. Then the torsion
submodule of $A$ is an $S$-module. Moreover, if $x$ is an element of infinite order in $A$ and $V(x)$ has infinitely many gaps or $L_{(x,v)}$ is cofinal with $u$, then the torsion submodule of $A$ is simply presented.

Now for the general case:

**Theorem 45.** Let $A$ be a Warfield. Then $A = \mathfrak{B} \otimes C$ where $\mathfrak{B}$ is balanced projective and the torsion submodule of $C$ is simply presented. In particular, the torsion submodule of $A$ is an $S$-module.

**Proof.** It is a straightforward application of Theorems 10 and 14, together with some manipulation of Ulm and Warfield invariants, to prove that $A = \mathfrak{B} \otimes C$ where $\mathfrak{B}$ is balanced projective and every element $x$ of a decomposition basis for $C$ is such that $V(x)$ has infinitely many gaps or $L_{(x,v)}$ is cofinal with $u$. Thus it is enough to show that $C_1$ is simply presented, let $T$ be simply presented so that $C \otimes T$ is simply presented. By Theorem 44, $C_1 \otimes T$ is simply presented and hence $C_1$ is simply presented.

In view of the preceding theorem, it is natural to ask whether we can decompose a Warfield module $A = \mathfrak{B} \otimes C$ so that $\mathfrak{B}$ is balanced projective and each element of a decomposition basis for $C$ has infinitely many gaps in its value sequence. However, this is not possible, and an easy counterexample can be obtained by adjoining a new element with value sequence $u, w = 1, w = 1, \ldots$ to the decomposition basis of the module in example 32 and leaving all the other details of the construction unchanged.

**References**


