MATH 541- Part II - summary

Material covered before the second exam. Write examples for each of the concepts defined below.

I. ALGEBRAIC TOPOLOGY

1. Covering spaces. \( p: E \rightarrow X \) is a covering space of \( X \) if every \( x \in X \) has an open neighborhood \( U \) such that \( p^{-1}(U) \) is a disjoint union of open sets \( S_i \) in \( E \) each of which is mapped homeomorphically onto \( U \) by \( p \). Such \( U \) are called evenly covered.

2. Application. We proved the path lifting lemma and the covering homotopy lemma. As an consequence we have that: \( p_*: \pi_1(E) \rightarrow \pi_1(X) \) is a monomorphism.

3. Covering transformations. For any covering space \( p: E \rightarrow X \), the group of covering transformations is the group of all homeomorphisms \( \phi \) of \( E \) which preserve the fibres, that is \( p \circ \phi = p \).

4. Theorem. Given a covering space \( p: (E, e_0) \rightarrow (X, x_0) \) with group of covering transformations \( G \). If \( E \) is simply connected and locally pathwise connected, then \( g \) is canonically isomorphic to \( \pi_1(X) \).

5. Example. \( \pi_1(\mathbb{RP}^n) = \mathbb{Z}_2 \).

II. HIGHER HOMOTOPY GROUPS

6. Higher homotopy. Consider the space \( X^I \) with the compact-open topology, and denote by \( \Omega_{x_o} \) the subspace of \( X^I \) consisting of all loops at \( x_0 \) (with base point the constant loop at \( x_o \)). We define for \( n \geq 2 \)

\[
\pi_n(X, x_0) = \pi_{n-1}(\Omega_{x_0}).
\]

7. Alternative definition. We showed that \( \pi_n(X, x_0) \) can be interpreted as homotopy classes of maps \( (S^n, s_0) \rightarrow (X, x_0) \).

8. Homotopy Sequence of a Fibration. To a fibration \( F \rightarrow E \rightarrow X \) there is associated a long exact sequence

\[ \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(X) \rightarrow \pi_{n-1}(F) \rightarrow \cdots. \]
9. **Important Example.** Using the Hopf Fibration

\[ S^1 \to S^3 \to S^2 \]

it follows that

\[ \pi_3(S^2) = \mathbb{Z}. \]

### II. Differential Topology

Review of Calculus:

10. **Inverse Function Theorem.** Let \( U \subset \mathbb{R}^n \) be open and \( f: U \to \mathbb{R}^n \) a \( C^r \) map \( r \geq 1 \). If \( Df_p \) is invertible, then \( f \) is a \( C^r \) local diffeomorphism at \( p \).

11. **Local Form of Submersions.** Let \( U \subset \mathbb{R}^m \) be open and \( f: U \to \mathbb{R}^n \) a \( C^r \) map \( r \geq 1 \). Let \( p \in U, f(p) = 0 \), and suppose that \( Df_p \) is surjective. Then there exists a local diffeomorphism \( \phi \) of \( \mathbb{R}^m \) at 0 such that \( \phi(0) = p \) and

\[ f \phi(x_1, \cdots, x_m) = (x_1, \cdots, x_n). \]

12. **Local Form of Immersions.** Let \( U \subset \mathbb{R}^n \) be open and \( f: U \to \mathbb{R}^n \) a \( C^r \) map \( r \geq 1 \). Let \( q \in \mathbb{R}^n \), such that \( 0 \in f^{-1}(q) \), and suppose that \( Df_0 \) is injective. Then there exists a local diffeomorphism \( \psi \) of \( \mathbb{R}^n \) at \( q \) such that \( \psi(0) = 0 \), and

\[ \psi f(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0 \cdots, 0). \]

Important definitions.

13. **Regular values.** Let \( f: M \to N \) be a \( C^1 \) map. We call \( x \in M \) a regular point if \( f \) is submersive at \( x \), otherwise we call \( x \) a critical point. A point \( y \in N \) is called a regular value if for all \( x \in f^{-1}(y), x \) is a regular point.

14. **Regular Value Theorem.** Let \( f: M \to N \) be a \( C^r \) map, \( r \geq 1 \). If \( y \in f(M) \) is a regular value, then \( Y = f^{-1}(y) \) is a \( C^r \) submanifold of \( M \). Moreover, \( \text{dim}(Y) = \text{dim}(M) - \text{dim}(N) \).

15. Some examples:

\( S^n \) is a \( C^\infty \) submanifold of \( \mathbb{R}^{n+1} \)

\( SL(n) \) if a \( C^\infty \) submanifold of \( GL(n) \).
III. Classification of Surfaces

We used the following two consequences of the Seifert–Van Kampen theorem:

16. Theorem. Suppose $X = U \cup V$ where $U$ and $V$ are open in $X$ and $U \cap V$ is pathwise connected. If $\pi_1(U \cap V) = 0$, then $\pi_1(X) = \pi_1(U) \ast \pi_1(V)$.

17. Theorem. Suppose $X = U \cup V$ where $U$ and $V$ are open in $X$ and $U \cap V$ is pathwise connected. If $\pi_1(V) = 0$, then $\pi_1(X) = \pi_1(U)/[\pi_1(U \cap V)]$, where $[\pi_1(U \cap V)]$ denotes the smallest normal subgroup of $\pi_1(U)$ containing $i_*(\pi_1(U \cap V))$.

18. Classification of Surfaces. Any connected compact surface is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes.