Exercise 1.4.3

Mathematics 683, Fall 2000

**Theorem 1.** Let $\mathcal{A}$ be an Abelian category. A complex $A$ in $\text{Ch}(\mathcal{A})$ is split exact if and only if $\text{id}_A$ is null homotopic.

*Proof.* Suppose that $\text{id}_A$ is null homotopic. Then there are maps $s_n : A_n \to A_{n+1}$ with $id = d \circ s + sc \text{id} rc$. If $\mathcal{A} = R - \text{Mod}$, then it is easy to show that $A$ is exact, since if $a \in A_n$ with $d_n(a) = 0$, then $a = \text{id}(a) = (d_{n+1} \circ s_n + s_{n-1} \circ d_n)(a) = d_{n+1}(s_n(a))$. Thus, $a \in \text{im}(d_{n+1})$. However, we give a proof for a general Abelian category. We have the diagram

\[
\begin{array}{cccccccc}
0 & \to & \ker(d_n) & \to & A_n & \xrightarrow{i_n} & A_{n+1} & \to & 0 \\
& & \uparrow{\varphi_{n+1}} & & \uparrow{id} & & \uparrow{id} & & \\
& & A_{n+1} & \xrightarrow{j_n} & A_n & \xrightarrow{d_n} & A_{n-1} & & \\
& & \downarrow{\pi_{n+1}} & & \downarrow{\pi_{n+1}} & & \downarrow{\pi_{n+1}} & & \\
& & \text{im}(d_{n+1}) & \to & A_{n+1} & & A_{n-1} & & \\
\end{array}
\]

with $i_n$ and $j_n$ monic, $\pi_{n+1}$ epic, and $j_n = i_n \circ \varphi_{n+1}$. The existence of $\varphi_{n+1}$ comes as follows: we have $d_n \circ j_n = 0$, since $0 = d_n \circ d_{n+1} = d_n \circ j_n \circ \pi_{n+1}$ and $\pi_{n+1}$ is epic. So, the definition of $\ker(d_n)$ gives a unique map $\varphi_{n+1}$ with $j_n = i_n \circ \varphi_{n+1}$. We show that $\varphi_{n+1}$ and $\pi_{n+1} \circ s_n \circ i_n$ are inverses. This will give $\text{im}(d_{n+1}) = \ker(d_n)$. We have

\[
i_n \circ (\varphi_{n+1} \circ (\pi_{n+1} \circ s_n \circ i_n)) = (i_n \circ \varphi_{n+1}) \circ (\pi_{n+1} \circ s_n \circ i_n) \\
= (j_n \circ \pi_{n+1}) \circ s_n \circ i_n = d_{n+1} \circ s_n \circ i_n \\
= (\text{id} - s_{n-1} \circ d_n) \circ i_n \\
= (\text{id} - s_{n-1} \circ (d_n \circ i_n)) \circ i_n = \text{id} \circ i_n = i_n.
\]

Since $i_n$ is monic, we conclude that $\varphi_{n+1} \circ (\pi_{n+1} \circ s_n \circ i_n) = \text{id}_{\ker(d_n)}$. Similarly,

\[
((\pi_{n+1} \circ s_n \circ i_n) \circ \varphi_{n+1}) \circ \pi_{n+1} = \pi_{n+1} \circ s_n \circ j_n \circ \pi_{n+1} \\
= \pi_{n+1} \circ s_n \circ d_{n+1} \\
= \pi_{n+1} \circ (\text{id} - d_{n+1} \circ s_{n+1}) \\
= \pi_{n+1} - \pi_{n+1} \circ d_{n+1} \circ s_{n+1} \\
= \pi_{n+1}.
\]
The equation $\pi_{n+1} \circ d_{n+1} = 0$ follows from $j_n \circ \pi_{n+1} \circ d_{n+1} = d_n \circ d_{n+1} = 0$ and that $j_n$ is monic. Thus, since $\pi_{n+1}$ is epic, we get that $(\pi_{n+1} \circ s_n \circ i_n) \circ \varphi_{n+1} = \text{id}_\text{im}(d_{n+1})$. Therefore, we have proved that $A$ is exact. It is then easy to see that $A$ is split since $d = d \circ \text{id} = d \circ (d \circ s + s \circ d) = d \circ s \circ d$.

Conversely, suppose that $A$ is split exact. Then $A$ is exact, and there are maps $t_n : A_n \to A_{n+1}$ with $d = d \circ t \circ d$. We need to produce maps $s_n : A_n \to A_{n+1}$ with $\text{id} = d \circ s + s \circ d$. Write $Z_n = \ker(d_n)$ and $B_n = \text{im}(d_{n+1})$. Since $A$ is exact, we may assume that $Z_n = B_n$. We have, for each $n$, an exact sequence

$$0 \to B_n \overset{i_n}{\to} A_n \overset{d_n}{\to} B_{n-1} \to 0$$

and since $d t d = d$, we have $d_n \circ t_{n-1} = \text{id}_{B_{n-1}}$ as $\text{im}(dt) \subseteq \text{im}(d) = B$. Thus, we may assume that $A_n = B_n \oplus t(B_{n-1})$. Define $s_n : A_n \to A_{n+1}$ by $s_n = t_n \oplus 0$. That is, the map $s_n : B_n \oplus t(B_{n-1})$ is the map obtained from the universal mapping property of coproducts arising from the two maps $t_n : B_n \to A_{n+1}$ and $0 : t(B_{n-1}) \to A_{n+1}$. It is clear from this definition that $s \circ t = 0$. We claim that $(d \circ t)|B_n = \text{id}|B_n$ and $(t \circ d)|t(B_{n-1}) = \text{id}|t(B_{n-1})$, which together will give $sd + ds = \text{id}$. More precisely, we claim that $(d \circ t \circ i) = \text{id}|B_n$ and $(t \circ d \circ t) = \text{id}|B_{n-1}$. We have a map $\pi : A_{n+1} \to B_n$ that satisfies $i \circ \pi = d$. Since $d \circ t \circ d = d$, we get

$$d \circ t \circ i \circ \pi = i \circ (\pi \circ t \circ i) \circ \pi$$

$$= i \circ \text{id} \circ \pi.$$

Since $i$ is monic and $\pi$ epic, we conclude that $\pi \circ t \circ i = \text{id}$. Thus, $(d \circ t)|B = \text{id}$. A similar argument gives $(t \circ d)|t(B_{n-1}) = \text{id}$. Therefore, $\text{id}_A$ is null homotopic. \hfill \square